AN ANALYSIS OF STUDENTS’ THINKING WHEN LEARNING DIFFERENTIATION AT ADVANCED LEVEL: A CASE STUDY OF MONTE CASSINO SCHOOL IN MASHONALAND EAST.

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Research project entitled “AN ANALYSIS OF STUDENTS’ THINKING WHEN LEARNING DIFFERENTIATION AT ADVANCED LEVEL: A CASE STUDY OF MONTE CASSINO SCHOOL IN MASHONALAND EAST” submitted by GUNGIRA LICKSON in partial fulfilment of the requirements for a degree of Master of Science Education in Mathematics.

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DECLARATION

I declare that An analysis of students’ thinking when learning differentiation at advanced level: A case study of Monte Cassino School in Mashonaland East, Zimbabwe is my own work and has not been submitted before for any degree or examination at any University and that the sources I have used have been properly indicated and acknowledged as complete references.

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DEDICATION

I dedicate this dissertation to two people who have been significant in my life: my father Nyamusoka Gungira, who showed me the value of hard work and perseverance and moulded me into the person I am today.
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(v)
ABSTRACT

The purpose of “An analysis of students’ thinking when learning differentiation at advanced level: A case study of Monte Cassino secondary school in Mashonaland East”, was to investigate and categorise students’ thinking when learning differentiation at advanced level. The study was a case study of a group of thirty five form five students at Monte Cassino Girls School in Mashonaland East province. Students’ thinking was analysed through content analysis. Data was collected through task based interviews. The interviews, which were done at regular intervals over a period of one month, during which the topic was taught, were structured to elicit students’ responses which revealed their thinking when learning differentiation. Thus, the research focused on identifying and categorizing students’ thinking using Vygotsky’s theory of concept formation as an analytical tool. The data collected was qualitatively analyzed. Since the study was premised on the constructivist perspective of learning, that learners construct knowledge on the basis of knowledge they already have, some of their constructions maybe full of misconceptions and errors. The study therefore revealed that when learning differentiation students display different forms of thinking which can be categorized as heap thinking, complex thinking and thinking in concepts. It was noted that heap thinking and complex thinking were prevalent, in particular associative and collection complexes. Although idiosyncratic constructions may appear meaningless, to the observer they are a critical component of concept acquisition. Errors and misconceptions when learning differentiation were found to be intertwined with other topics such as algebra, substitution and factorisation. The results of the study indicated that the teachers’ awareness of students’ thinking in learning differentiation is critical in developing appropriate pedagogical content knowledge. The understanding of the learners’ thinking affords teachers to plan effective teaching strategies to help students understand differentiation. The study therefore recommends that teachers should focus more on students’ thinking when teaching as illustrated by Vygotsky’s theory of concept formation.
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Chapter 1: The problem and its context

1.0 Introduction

There have been widespread demands that mathematics instruction should be reformed so that students learn mathematics with understanding by actively participating in tasks that incorporate important mathematics (National Council of Teachers of Mathematics, NCTM: 1989, 1991). According to Fennema, Carpenter, Franke, Levi, Jacobs & Empson (1996) research findings suggest that developing an understanding of children's mathematical thinking can be a productive basis for helping teachers to make the fundamental changes called for in current reform recommendations.

Although there is consensus that teachers' knowledge is a major determinant of mathematical instruction and learning (Fennema and Franke, 1992), there is however little agreement and even less evidence about what knowledge will enable teachers to teach so that students learn mathematics with understanding. (Fennema et al. 1996).

As cited by Fennema et al, there is evidence to support that changing teachers' knowledge may initiate change in teaching practice (Clark & Peterson, 1986, Fennema & Franke, 1992, Putnam, Hampert and Peterson, 1990) and that knowledge of children’s thinking is a powerful influence on teachers as they consider instructional change. It is therefore imperative that in learning mathematics, students must be taught concepts for understanding. To achieve this, the mathematics teacher must execute his/her fundamental role of analyzing the students’ thinking and work. This important role involves interpreting student explanations and making sense of what pupils are saying, determining the mathematical validity of the students’ strategy, solution or conjecture as well as figuring out what students know and do not know, as
well as what conceptual knowledge connections are missing or are fragile (Ma, 1999). Creating mathematical classrooms where mathematical thinking is central is critical as it complements the other teachers’ roles, which according to the National Council of Teachers of Mathematics (1991), include posing worthwhile mathematical tasks, orchestrating stimulating mathematical discourses and thoughtful planning and reflecting on their teaching.

In order to analyze students’ mathematical thinking teachers need to have knowledge of students’ mathematical understandings, which involves both conceptual understanding and misunderstandings. Misunderstandings are usually categorised as errors and overgeneralisations. In most cases, teachers view students’ errors as ignorance or carelessness (Radatz, 1980), rather than a form of thinking, which is inevitable as students construct new mathematical knowledge. According to Cobb (1988) and Maher & Davis (1990) cited in Crespo (2000), when teachers fail to listen to or do not understand their students’ thinking they tend to dismiss thinking that results in wrong answers and impose their own formalised constructions on to the students. Studies done by Carpenter, Fennema, Peterson & Carey (1988) cited in Fennema et al (1996) showed that teachers have informal, although unfocused, knowledge about students’ mathematical thinking. They therefore, often recognized some distinctions among students’ mathematical problems and can identify many of the primary strategies used by students, but they seldom relate critical problem dimensions to children’s solutions or problem difficulty nor does knowledge of their students’ thinking play a critical role in planning instruction.

My observation as a mathematics teacher is that teachers do not have time in their classrooms to analyse students’ errors to understand students’ thinking processes and structure their teaching in a way that helps the students understand better or avoid
these errors and misconceptions. They seem to concentrate more on completing syllabi and drilling the students to pass examinations. However, according to Borasi (1987), errors can be a powerful tool to diagnose learning difficulties and consequently direct remediation. According to him, analyzing student errors (with the hope of understanding their thinking) have provided valuable contributions to mathematical education, such as an increased awareness of individual differences and difficulties in learning mathematics, and the realisation of the inefficiency of remediating errors by simply explaining the same topic over again or assigning additional practice experiences. Borasi further noted that most studies in error analysis still share a rather limited conception of errors, one in which errors are seen as a signal that something has gone wrong in the learning process and that remediation is called for. In view of these observations, the present study used Vygotsky’s theory of concept formation as a framework to analyze students’ thinking as they learn differentiation at advanced level.

1.1 Background to the study

Students’ achievements in public examinations in mathematics have always been a concern for both educators and policy makers in Zimbabwe. Moyo’s (2003), as cited in Nyikahadzoyi (2006), analysis of Zimbabwe learners’ performance in public examinations at all levels from 1980 to 2000 shows that learners’ performance was low in mathematics compared to performance in other subjects. In fact the analysis noted that the pass rate has been declining over the years instead of improving. This observation is supported by the Ministry of Education (1995) in the Mathematics Teacher’s Handbook, a product of the Associate Teacher Programme (Curriculum development unit). Nyikahadzoyi cited three factors, among several factors, that have affected learners’ performance as poor curricula material (Jaji and Hodzi, 1980),
inappropriate mathematics education preparation programmes (Ball, 2000) and teachers’ inability to plan and organize rich learning experiences for learners (Jaji, 1990). Faced with such a dilemma, mathematics education has gained significant momentum as a national priority and important focus of school reform. Central to raising students’ achievement in mathematics is improving the quality of mathematical teaching (McGraner, VanDerHeyden and Holdheide, 2011).

The reform movement, according to the Education Alliance (2006) was a response to the failure of traditional teaching methods and is driven by the impact of technology on curriculum and the emergence of new approaches to the scientific study of how mathematics is learned. Central to these new approaches is constructivism, which asserts that mathematical knowledge is socially constructed. Evidence from research indicates that individual students learn mathematics by constructing mathematical meaning themselves through incorporating what they already know (Davis, Maher and Nodding, 1990 cited in Anku, 1996) and that this individual construction is mediated by social context within which students learn (Yackel, Cobb, Wood, Wheatley &Merkel, 1990). Yackel et al further argued that teachers must take responsibility for helping students verbalise their mathematical ideas in a mathematically meaningful way. This can therefore be achieved if teachers understand students’ mathematical thinking patterns.

According to Harel (1998, in press) (as cited in Harel & Sowder, 2005), the Duality principle help teachers to appreciate the link between students’ thinking when learning mathematics and their understanding of mathematical concepts. He argued that students’ ways of thinking impact their ways of understanding mathematical concepts, and conversely, that how students come to understand mathematical concepts influences their way of thinking. Being aware of and eliciting common
misunderstandings and drawing students’ attention to the misunderstandings can therefore be a valuable teaching technique (Griffn and Madgwick, 2005 as cited in Tobey & Minton, 2010). A student’s way of thinking, both good or bad, influences one’s further ways of understanding mathematical concepts. Most of us want our students to understand information that is presented to them. This understanding is not simply remembering mathematical concepts or being able to follow procedure (Idris, 2009). The process of diagnosing students’ understandings to understand their thinking and then making instructional decisions based on such information is therefore important in increasing students’ mathematical knowledge, and therefore can help address the issue of low achievement in Mathematics.

The knowledge of students’ mathematical thinking constitutes part of the teachers’ pedagogical content knowledge (PCK), which is a very important component of teacher knowledge. PCK, according to Shulman (1986) consists of the ways of representing and formulating the subject that makes it comprehensible to others. Nyikahadzoyi, Julie, Mtetwa and Torkildsen (2008) acknowledged that since Shulman many educators have attempted to elaborate pedagogical content knowledge, an indication that it is a critical component of the teacher’s knowledge in teaching mathematics. There is consensus among educators that teachers need a special kind of knowledge for teaching, hence the need to analyze students’ mathematical thinking as it makes a contribution to this special kind of knowledge. Shulman (1986) cited in Tobey & Minton (2010) regards knowledge of potential difficulties as being at the centre of a teacher’s required pedagogical knowledge. The need of PCK cannot be overemphasised as argued by Ball (2000:245) as cited in Verwey (2010) when he said:
“Being able to see and hear from someone else’s perspective, to make sense of a student’s apparent errors or appreciate a student’s unconventionally expressed insight requires this special capacity to unpack one’s own highly compressed understandings that are the hallmark of expert knowledge”.


Although Zimbabwean teachers get to know about students’ thinking patterns as indicated by misunderstandings students display in mathematics from examiner reports, noted misunderstandings are not extensively analysed to help teachers design rich instructional programmes. Having gone through examiner reports from 1997 to 2004, I noted that errors and misunderstandings are just mentioned without an in depth analysis by way of identifying the cause of such thinking patterns and acknowledging that they are a form of thinking in learning mathematical concepts. The majority of teachers in Zimbabwe do not have any forum to discuss these identified errors. Although efforts have been made in Zimbabwe to facilitate programmes that enable teacher organised workshops, financial challenges have hampered progress. The Science Education Inservice Teacher Training (SEITT) and Associate Teacher Programme are examples of such initiatives, established to help alleviate teachers’ challenges. The SEITT workshop reported by Nyaumwe (2008) had only sixteen participants who had the opportunity to deliberate on students’ thinking (which they
termed alternative conceptions) in the form of students’ errors to draw professional insights from students’ conceptions. Very few teachers have such an opportunity in Zimbabwe. Such a scenario prompted the current research to analyse students’ mathematical thinking in learning differentiation at advanced level.

According to Radatz (1980) most students’ errors are not due to unsureness, carelessness or unique situational conditions, as was assumed at the beginning of the behaviourist theory of learning. Behaviourism assumed that behaviour can be studied in a systematic and observable manner with no consideration of internal mental states. This school of thought suggests that only observable behaviours should be studied since internal states such as cognition, emotions and moods are subjective. According to Olivier (1992) (as cited in Verwey, 2010), behaviourists, assume that new information is isolated from previously learned knowledge, and so errors are not important in learning. Only correct knowledge from successful procedures has significance for behaviourism, while errors are regarded as negative with no pedagogical value (Henze, 2005, Leu & Wu, 2005 in Verwey).

On the contrary, Radatz rather pointed out that errors are a result or the product of previous experiences in the mathematics classroom. In recent years there has been a shift in education to moving towards a more constructivist approach to the teaching of mathematics (Solso, 2009). While behaviourism emphasizes passive absorption of observable behaviours, constructivism asserts that individuals assimilate new information and incorporate it into their already existing schema and subsequently make their own sense of mathematical concepts. Contrary to constructivism, behaviourism believed that learners’ misconceptions and related errors can be overruled by teaching correct procedures (Gagne, 1983 cited in Verwey, 2010); therefore learning takes place when correct procedures are enforced. Based on the
present state of error research (1980), Radatz therefore, concluded that, students’ errors: are determined in an unconcerned way and not planned by teachers, are persistent and will last for several school years unless the teacher intervenes pedagogically. They can be analysed and have their causes identified through certain difficulties experienced by students while receiving and processing information in the mathematical learning process or from effects of the interaction of variables acting on mathematics education (e.g. teachers, students, curriculum, academic environment etc)

Analyzing these errors can therefore be a good way to help teachers understand their students thinking. Borasi (1987) however noted that most studies share the view that errors are a sign of something that would have gone wrong in the learning process and that remediation is required. It is therefore important to note that whatever form of understanding shown by students, it is achieved through some form of thinking. The current research therefore used Vygotsky’s theory of concept formation, which explains students’ thinking in learning concepts. In the study, the theory was used in the classroom while teaching differentiation to analyze how students’ think; in particular when they showed misunderstandings commonly identified as mathematical errors. The research took a different dimension from that of viewing errors as signalling the need for remediation by analyzing students’ thinking patterns in learning differentiation at advanced level using Vygotsky’s theory as an analyzing tool.

Mathematics education reform has its focus on problem solving, mathematical reasoning, justifying ideas and independent learning of new ideas. This can be achieved by giving open ended mathematical tasks (Julie and Torkildsen, 2008). The solution procedures for such tasks are not immediately identifiable and learners invent their own solution methods. Such solution procedures might appear as idiosyncratic
and display features that may be deemed as non mathematical. This focus on reforms in mathematics education prompted me to take this dimension of analyzing students’ thinking in learning differentiation with the aim of further increasing awareness on the need to make students’ voices heard in the classroom and that teachers structure their teaching to help the students better. Thus it is anticipated that the results of this study will make a small contribution to the teachers’ pedagogical content knowledge in teaching differentiation. Such knowledge would enable teachers to fulfil their teaching roles and hopefully improve on the teaching of differentiation. This is hoped to make a contribution on students’ achievement in Mathematics. Recent work in assessment also reflects a growing sensitivity to the importance of students’ misunderstandings.

Recent research suggests that analysis of students’ thinking is seen as a resource that can help teachers make informed decisions in their classrooms and improve their practice (Doeer, 2006; Kazemi & Franke, 2004; Stenberg, Empson & Carpenter 2004, cited in Peng & Song, 2008), so inquiry into students understandings, which impacts their mathematical thinking, has also been seen as a means to teacher development. Peng and Song asserted that the link between error analysis, which is an important way to inquire into students’ thinking, and mathematics teachers’ professional development is largely unexamined, particularly at secondary level. In addition, the current trend is to ensure that instructional programs enable students to understand and use mathematics in a technological world (Swan, 2001) and ever widening range of activities. Booker (2011) noted that those who lack the ability to think mathematically will be disadvantaged, unable to participate in high level work. In addition research on an inquiry into students’ thinking seem to have concentrated on elementary and tertiary levels, leaving a gap at advanced level (Peng and Song, 2008), which this research sought to bridge.
1.2 Statement of the problem

One reason that has been cited as a cause for poor performance in Mathematics in Zimbabwe is teachers’ inability to plan and organise rich learning experiences for learners (Jaji, 1990 as cited by Nyikahadzoyi, 2006). In view of this observation, this research sought to make contributions in this regard by analysing how students think as they learn the topic of differentiation at advanced level. Analysis of students’ thinking was done by using Vygotsky’s theory of concept formation as a lens to analyze how students think. The knowledge gained from this research is hoped to contribute to the teachers’ Pedagogical Content Knowledge (Shulman, 1986) which is critical in the teaching of mathematics. This in turn would enable teachers to plan rich and meaningful learning experiences for their learners, an act that is anticipated to make a small contribution in the attempt to improve Mathematics pass rates at advanced level in Zimbabwe.

1.3 Research Question

The research was guided by the following research question:

What are the different ways of thinking displayed by students when learning differentiation?

The research question was broken down into sub-questions centred on the students’ thinking:

- What form of heap thinking is used by A Level students when learning differentiation?
- What are the various complexes displayed by advanced level students when learning differentiation?
• What are the other characterizations of students thinking in learning differentiation at advanced level?

1.4 Purpose of study

The purpose of the present study was to identify, examine and illustrate how students think when learning differentiation at ‘A’ level. The aim of the study was therefore to understand how students think as they construct their mathematical knowledge when learning this topic. It is therefore the objective of this study to analyse students’ thinking by using Vygotsky’s theory of concept formation to get insights on students’ thinking processes and categorize them. It is hoped that the results of this study will be helpful to teachers when teaching differentiation at ‘A’ level, thereby help sensitize them to listen to the pupils’ voices in the classroom when they analyze students’ work, whether written or verbal.

1.5 Importance of the study

Borasi (1987) observed that a review of the mathematical education literature shows that researchers and teachers in this field have seriously considered students’ mathematical misunderstandings, commonly called mathematical errors. However, most of them have been restricted to only acknowledging the existence of learning process problems rather than understanding children’s thinking patterns in concept formation as manifested in the errors they commit when learning as forms of thinking. The notion of specialized knowledge base for teaching has been in existence for more than twenty years (McGraner, VanDerheyden & Holdheide, 2011). Shulman (1986) was the first to identify a specialised form of knowledge necessary for the practice of effective teaching which he called pedagogical content knowledge (PCK). He defined this knowledge as the knowledge and means of; “representing and formulating the
subject matter that makes it comprehensible to others” (Shulman, 1986:9). Shulman argued that content-absent pedagogy is problematic to the classroom teacher who relies on content knowledge to deliver and advance student thinking.

Mathematics teachers must not know only the content they teach, but also how students’ knowledge is developed and structured, how to manage internal and external representations of concepts, how to make students mathematical understanding visible and how to diagnose students’ misunderstandings, correct them and guide them in reconstructing complex conceptual knowledge of mathematics (Ball, Lubienski & Mewborn, 2001; Cohen & Hill, 2000; Darling-Hammond, 1999; Fennema & Franke, 1992 cited in McGramer et al., 2011,p.7)

The preceding citations are a testimony of the importance of PCK, hence the need to analyse students’ thinking patterns in the learning of differentiation at advanced level. The knowledge gained from the study will therefore contribute to the PCK needed to teach differentiation which, Shulman believed is important in the teaching of mathematics. Other researchers have also indicated that increasing teachers’ knowledge of students thinking helps them design better instructional tasks, ask better questions during lessons and assist individual students more effectively (Schoenfeld, 2005; Borasi & Fonzi, 2002; Crespo, 2000; Confrey, 1991). Fennema et al (1996) argued that the analysis of students' thinking can be thought as scientific knowledge as defined by Vygotsky (1962) that provides a basis for teachers to interpret, transform and reframe their informal knowledge of students’ mathematical thinking. Since one way to improve mathematical instruction and learning is to help teachers understand the mathematical thought processes of their children, Fennema et al and Cobb et al (1990) believe that such knowledge is not static and acquired outside the classrooms in workshops but is dynamic and ever growing and can probably only be acquired in
the context of teaching mathematics. They further asserted that teachers can use misunderstandings that become apparent during classroom discussion as instructional leverage to make explicit the key concepts and tools of inquiry needed to understand and apply mathematical principles. Schifter (2001) also argued that the PCK helps teachers develop critical teaching skills such as; ability to attend to the mathematics in what students say and do, ability to assess the mathematical validity of students’ ideas (even if the work looks non-standard) and ability to listen for the sense in students’ mathematical thinking.

Schifter argued that once one is alert to the mathematical possibilities in students’ thinking, one can often find the correct core of a correct mathematical approach in something that produces an incorrect answer. Thus it becomes imperative that with an increased awareness on students’ thinking patterns in learning, then it is hoped that this will translate into improved teaching and therefore improved mathematics results.

The current study is therefore in line with the initiative of the Associate Teacher Programme, whose objectives among many others include helping teachers to cope with specific thinking patterns and sensitize teachers that they are not just misunderstandings requiring remediation, but forms of thinking when learning differentiation (Curriculum development unit, 1995), although suggestion of suitable methodologies would be an area for another study.

Differentiation can be applied to help solve many real world problems. Application and modelling seem to be a key skill emphasized in many school curriculums including Zimbabwean curriculum. It is a skill that enables application of concepts and skills to solve problems of variety contexts within and outside mathematics. Stewart (2004)
cited in Makonye (2011) explained that mathematical modelling provides learners with the means to analyse their world mathematically. Tangents and normals for example, are important in physics (e.g. Forces acting on a car negotiating a corner), curvilinear motion, related rates of change and Raphson Newton’s method of solving equations (computers use iterative methods to solve equations), just to mention a few, are situations that require the use of differentiation of functions. With knowledge on students’ thinking, it is hoped that the research will add to existing pedagogical content knowledge which in turn is hoped to contribute to the development of highly skilled scientifically and technologically based manpower, which requires a strong grounding in differential mathematics (Ministry of Education Singapore, 2006; Mathematics Teachers’ Handbook, Zimbabwe, 1995). Learners at advanced level in Zimbabwe are expected to develop capacity to use principles of differentiation to determine rates of change of a range of non linear functions solve simple optimisation problems and apply differential model.

According to Golden (n.d) in his lecture “what is calculus about and why should I study it”, the story of calculus began in the late 1600s with the revolutionary results of Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716) in understanding motion and rates of change. Some of the underlying concepts, such as infinite and the infinitesimal were however thought about and articulated by the ancient Greeks, Zeno and Archimedes well over 200 years ago. Golden proposed that with historical hindsight it can be concluded that the development of calculus is certainly one of the greatest intellectual achievements of the past two millennia. Bishop Berkeley cited by Makonye (2011), who was an Irish bishop and philosopher in the 17th century, provides a good example of how identification of an error or misunderstanding can fuel the advancement of knowledge. His contribution to mathematics and calculus in particular
was his attack on the logical foundation of the calculus as developed by Newton and Leibnitz, the principle of infinitesimal leading to the limit, which forms the basis of calculus.

Golden argued that calculus provides the language and basic concepts used to formulate most of the fundamental laws and principles of various disciplines throughout the physical, mathematical, biological, economic, social sciences and mechanical or electrical engineering. It serves as an inescapable gateway to all higher level courses. It provides a framework to analyse the most basic and essential properties of functions. According to Golden, the reason calculus has monopoly in describing our world is that almost any quantitative modes of a physical, chemical, biological, engineering, financial or industrial system involves the use of a functions. Any analysis of these functions used in understanding or predicting how it evolves in time will invariably involve calculus in the form of differentiation. It is for these reasons that students who intend to major in fields of sciences, engineering, medicine and business are required to take differentiation and integration topics, and Zimbabwean curriculum is not an exception.

It therefore cannot be overemphasised that the topic of differentiation forms the list of important topics in Zimbabwean curriculum at advanced level. Neil and Shuard (1986) in Makonye (2011) termed calculus the language of higher mathematics, sciences and technology. Makonye viewed it as the epicentre of scientific knowledge making differentiation a very important topic in any curriculum. Although Makonye acknowledged that many researchers have reported misunderstandings when learning differentiation, the studies were however not done in Zimbabwe. In Zimbabwe common students’ misunderstandings when learning this topic have been highlighted
in examiner reports, though it is not clear if an analysis has been done to get insights on students’ thinking patterns and characterize them.

1.6 Definition of terms

In this study the following definitions were used in the context of Vygotsky’s theory of concept formation to analyse students’ thinking.

1.6.1 Heap thinking

This is thinking in which a student lists ideas or objects together as a result of idiosyncratic association. It is a form of thinking that forms an initial stage in the child’s development of the process of generalization, though the child groups together disparate and unrelated objects that are linked by chance in the child’s perception. In this research a child generalizing from the sequencing of questions or parts of a question is using heap thinking.

1.6.2 Complex thinking

According to Vygotsky (1986) cited in Berger (2004) this a form of thinking when individual objects are linked in the child’s mind not only by his/her subjective impressions but also by bonds actually existing between these objects. Tuomi (1998) defines a complex as a concrete grouping of objects connected by factual bonds. Berger concurs when she pointed that the bonds are factual and derive from experience rather than being systematic or based on logic. In the current research a child who generalize from a single example or from one attribute of a result(nucleus) or familiar signs, and apply the generalized result to a new problem or concept is using complex thinking.
1.6.3 Concept

Berger (2004) defined a concept as a mathematical idea which has its internal links and external links consistent and logical. This means that the links between the different properties and attributes of the idea as well as the links of the concept to another are all consistent and logical. A concept is therefore an element of thought, a mental construct that represents a class of objects. Concepts consist of a series of characteristics that are shared by a class of individual objects. These characteristics which are also concepts, allow students to structure their thoughts and to communicate. Vygotsky categorized these concepts into spontaneous concepts, which derive directly from students’ experience with the world and scientific concepts, which a student acquires through their exposure within a structured school environment with their development being deductive.

1.6.4 Understanding and Thinking

According to Wiggins & McTighe (2005) understanding is about transfer that requires the ability to transfer what we have learned to new and sometimes confusing settings. When we understand we can create new knowledge and arrive at further understanding if we have learned with understanding some key ideas and strategies. Perkins & Blythe (1994) cited in Idris (2009) define understanding as being able to explain, finding evidence and examples, generalizing, applying and representing a topic in a new way. Teaching mathematics for understanding is therefore to help students develop how to think and how to make decisions. Skemp (1976) distinguished understanding into instrumental understanding, that which deals with procedural knowledge involving the use of rules and procedures without reasons why they work and relational understanding, that involves knowing what to do and why it
works that way and therefore constituting conceptual knowledge. According to Harel and Sowder (2005) the particular meaning students give to a term, sentence or text, the solution they provide to a problem or justification they use to validate or refute an assertion are ways of understanding. On the other hand students’ general theories, implicit or implied underlying such actions are ways of thinking. Thinking involves students identifying and posing problems as well as selecting and applying appropriate strategies to find solutions. It involves conjecturing and proving, applying and verifying, generalizing and using mathematical models, communicating ideas and solutions as well as reflecting.

1.6.5 Misunderstandings

Cobb, Wood and Yackel (1990) defined misunderstanding as a mapping of a working idea in a plausible but incorrect way in a new situation. Misunderstanding is not ignorance. Paradoxically, a student has to have knowledge and the ability to transfer in order to misunderstand things. Misunderstanding signifies an attempted and plausible but unsuccessful transfer, which therefore involves thinking.

1.6.6 Error

According to Tobey & Minton (2010) an error refers to systematic use of inaccurate and inefficient procedure or strategy. This type of error pattern indicates non understanding of concepts. On the other hand Verwey (2010) defined an error as any response, contribution or question by a learner that puzzles or that contrasts with what is expected. Errors have been metaphorically referred to as symptoms of a disease, the misconception (Borasi, 1988). To Leu & Wu (2005) in Verwey (2010), errors are examples of misconceptions or irregularities in learners’ thinking and reflect the level of conceptual understanding of mathematics. Thus an error in this research was taken
as any deviation from the expected socially sanctioned mathematical procedures and results in differentiation. Such deviations were viewed as showing forms of thinking by the students in their attempts to differentiate functions.

1.6.7 Misconceptions

According to Askew and William (1995) cited in Swan (2001) one of the most important findings of mathematics education research carried out in Britain over the last twenty years has been that all pupils constantly ‘invent rules’ to explain the patterns they see around them. While many of these rules are correct, they may only apply in a limited domain. When pupils systematically use incorrect rules or use correct rules beyond their proper domain of application, then it becomes a misconception leading to errors. Nesher (1987) believes that the notion of misconception denotes a line of thinking that causes errors all resulting from an incorrect underlying premise, while Erlwager (1975) looks at a misconception as a knowledge structure that is activated in a variety of contexts, is stable and can be resistant to change and competes with the accepted cannons of scientific knowledge.

Makonye (2011) summed these views about misconceptions by pointing out that misconceptions could be generalisations of earlier acquired knowledge, valid knowledge wrongly applied to an extended domain. That is, misconceptions result in misapplication of algorithms and rules in domains where they are not applicable. For example if students are given the question: “Find the minimum or maximum value of the function $f(x)$” and they equate $f(x)$ to zero and perform algebra on it instead of equating $f'(x)$ to zero, then they would be showing a misconception. This confirms the view that misconceptions are also a form of understandings implying a particular way of thinking and therefore critical for this study.
1.6.8 **Analysis of Thinking**

Analyzing students’ thinking involves the in-depth examination and discussion of selected artefacts of students’ mathematical activity (Borasi and Fonzi, 2002). In this research, analysis of students’ thinking involved examining students’ written work, students’ class presentations and selected students’ artefacts. This analysis aimed to identify error patterns to identify difficulties that students may have with facts, concepts, strategies, procedures, and misconceptions as indicators of students’ thinking for characterization using Vygotsky’s theory of concept formation.

1.7 **Limitations of the study**

Although a case study research design chosen for this study provided data for in-depth analysis of student mathematical thinking, it would have been ideal to do a longitudinal study over two years to follow up on students thinking up to form six. This would have provided data to check if students still display heap and complex thinking after their initial introduction of the topic of differentiation in form five. In addition, since a case study involves one item or thing, in the case of the current study, a single school and one researcher collecting data, this can lead to bias in data collection which can influence results more than in other different designs. Moreover, some members of the scientific community frown upon the case study design because researchers using it often violate the principle of falsification. In modern post-positivist scientific thought the researcher takes the role of the disinterested observer; he or she has no vested interest in whether the research turns out one way or the other (Guba & Lincoln, 1994), which may not be possible in a case study design. It would also have been ideal to include more schools that would include boys since the current study involved girls.
only. The study was however limited to this design due to limited time and resources as the research had to be done and concluded in six months.

1.8 Delimitations of the study

Although the Duality Principle was assumed, the current research focused analyzing students’ thinking when learning differentiation; it was not about students’ understanding of mathematical concepts. The study was restricted to the topic of differentiation, although other topics were mentioned in the analysis of students’ complex thinking as they learn this topic. The analysis of students’ thinking was done by analyzing what students said during lesson discussions or interviews and what they wrote in their exercise books over a six-week period when the topic was taught. A case study of girls only was used to collect data. The categorization of students thinking was done using Vygostky’s theory of concept formation. Heap and complex thinking used by students when learning other topics were only analysed in the context of differentiation. The current study was conducted at Monte Cassino Girls’ school in Mashonaland East Province with the researcher the only one involved in the collection of data.

1.9 Assumptions

Two assumptions were made in carrying out the current research. It was assumed that whenever students attempt to do assigned tasks, whether their solutions are written or verbal they are engaged in thinking. This thinking can result in correct or wrong
answers. It was also assumed that the Duality Principle best explains student thinking in learning mathematical concepts. The assumption is that students’ understanding of concepts influences their thinking pattern and the thinking they develop determines how they understand mathematical concepts.

1.10 Ethical considerations

In carrying out the current research, permission was sought from the responsible authority to do the research at the Roman Catholic school. The consent of the students was sought before the research was done. The students were alerted that their participation was voluntary and that the research was purely academic, and the results meant to be used to assist other teachers to improve the teaching of the topic of differentiation.
Chapter 2: Theoretical Framework: Vygotsky’s theory of concept formation

2.0 Introduction:

This chapter focuses on the theoretical framework which forms the basis of the current study and therefore helped to focus the literature review.

2.1 Roles of theoretical frameworks

Newman (1991) cited in Nyikahadzoyi (2006) defines a theoretical framework as an orientation or a way of looking at a social phenomenon or construct. In this study Vygotsky’s theory of concept formation was used as an analytical tool for analysing students’ thinking in learning differentiation. According to Nyikahadzoyi (2006) theoretical frameworks are used to guide empirical inquiry as they provide a structure for an inclusive explanation of empirical phenomenon, its scope and how we should look at and think about the construct. In the current research the construct is students’ mathematical thinking. The framework therefore provided an orderly scheme for classification and description of students’ thinking in learning differentiation. The use of Vygotsky’s theory in the current study is consistent with what Nyikahadzoyi suggested when he argued that theoretical frameworks do not only direct the researcher to the important questions, but can also be used as analytical tools to make sense of research data. Thus the theory was valuable in classifying and explaining students’ thinking in this study.

2.2 Theoretical Assumptions

Vygostsky’s theory of concept formation can be taken to be a powerful framework to describe advanced level mathematics learners’ thinking as they learn new mathematical concepts. Berger (2005) argued that the theory is able to bridge the divide between an individual’s mathematical knowledge and the body of socially
sanctioned mathematical knowledge. It is built around the social constructivist philosophy which believes that mathematical concepts are explicitly constructed as an individual interacts with the environment. As noted by Ball (1990) there has been a shift in recent years in education in moving towards a more constructivist approach to the teaching of mathematics. Constructivists believe that knowledge only exists in the shared constructions of the learner and knowledgeable others (Yackel, Cobb, Wood, Wheatley and Merkel, 1990). Such a view is shared by other researchers who believe that mathematical understandings are both a personal and cultural enterprise (Schliemann & Carraher, 2002) and that mathematical learning is an interactive as well as constructive activity (Cobb, 1998). It is personal in so far as it entails invention and rediscovery. It is cultural because it relies on conventional symbol system and social contexts. Since Vygotsky’s theory of concept formation is consistent with these constructivist tenets, it therefore offers a powerful framework within which to analyze students’ mathematical thinking. Vygotsky’s theory is therefore rooted in the following basic assumptions that; Mathematics is a socially sanctioned body of knowledge, Mathematics ideas exist in the social world, through the use of mathematical signs (objects) in socially regulated discourse and activities the student develops a personal meaningful concept that is also meaningful to the wider mathematical community and learning is social regardless of the way it occurs. It does not need to be interactive to be social (Sfard, 2003 cited in Berger, 2004a)

In order to understand the theory, Berger (2005) advocated the need to understand how Vygostky used the term ‘word’, which he regarded as embodying a generalisation and hence a concept. According to Vygostky (1986) cited in Berger, a functional use of the word, or any other sign, as a means of focusing one’s attention, selecting distinctive features, analysing and synthesising the features, plays a central role in
concept formation. When Vygotsky considered concept formation as being an interactive and dynamic activity, it means that the formation occurs through the interaction of language and other signs (Wellings, 2003). A child does not spontaneously develop concepts independent of their meaning in the social world, a meaning is given through interactions with adults. Similarly, in mathematics a student is expected to construct a concept whose meaning and use is consistent with its use in mathematics community (Yackel et al, 1990; Maher and Noddings, 1990)

Vygostky’s theory of concept formation is a cognitive development process which explains how students construct their knowledge when learning mathematical concepts. Wertsch (1985) cited in Tuomi (1998) summarised Vygotsky’s theoretical framework as three core themes, that are consistent with the assumptions identified by Sfard which are; a reliance on a genetic or developmental method, the claim that higher mental processes in the individual have their origin in social processes as well as the claim that mental processes can be understood only if we understand the tools and signs that mediate them.

Vygotsky therefore argued that concept formation consists of two parallel processes; organising discrete elements into groups and abstraction of some aspects of attended phenomenon (Vygotsky, 1986 as cited in Tuomi, 1998). He argued that we use cognitive tools to control our thinking, tools that are a product of social, cultural and historical processes. According to Wellings (2003) Vygotsky describes concepts as being part of a system of representation encompassing both levels of abstraction and degrees of relatedness to a reality constructed of other concepts. This multidimensional representation supports the development of interlinked hierarchies that rely on existing concepts to facilitate the initiation of new concepts. The theory fits well into the constructivist’s philosophy of learning mathematics which asserts that
students approach new mathematical tasks with prior knowledge, assimilate new information and subsequently construct their own meaning. Vygotsky’s theory of concept formation therefore became a good framework for this study to analyze students’ thinking.

2.3 Vygostky’s stages of concept formation

Vygostky advocated a three stage-process in the formation of concepts that is developmental but not age related, which can be summarised as;

**Figure 1**: Vygotsky’s developmental theory of concept formation

![Diagram of Vygotsky's Developmental Theory of Concept Formation](image)

2.3.1 Heap stage

During this stage the child groups together objects or ideas which are objectively unrelated. Grouping occurs due to chance, circumstance or subjective impressions in child’s mind. According to Tuomi (1998) the child sorts objects into heaps based on syncretic organization of the child’s visual field. This form of thinking, termed syncretic thinking, is the initial stage in the child’s development of the process of generalisation. In the mathematical domain, a student is using heap thinking if she/he associates one mathematical sign with another because of, for example, layout of the page (Berger, 2005). The syncretic images play the role of concepts. Wellings (2003) argued that the children at this stage relies on their own perception to make sense of objects or items that appear to them unrelated. Students at this stage therefore can rely on trial and error activities and the organisation of their own visual field and perceptions of
time and space to create their own subjective relationships between objects. This results in children mistaking their egocentric perspective as reality.

2.3.2 Complexes Stage

This stage is characterised by ideas linked in the child’s mind by associations or common attributes which exist objectively between ideas. This is important as it allows the learner to think in coherent terms to communicate through words and symbols about a mental activity. It is this communication with more knowledgeable others which enable the development of a personal meaningful concept whose use is consistent with its use by the mathematical community (Berger, 2005; Yackel et al, 1990). According to Vygotsky (1986) cited in Berger (2005):

> Complexes corresponding to word meanings are not spontaneously developed by the child: The lines along which a complex develops are predetermined by the meaning a given word already has in the language of adults.

Vygotsky (1986) cited in Wellings (2003), asserts that these complexes’ major role is to establish bonds and relationships, which begins with the unification of scattered impressions by organizing discrete elements of experience into groups. It is during thinking in complexes when learners begin to isolate different attributes of ideas or objects as well as organising ideas with particular properties into groups, creating a basis for later more complicated generalisations. Learners do not use logic at this stage, but some form of non-logical or experiential association. Whereas in heap thinking a child mistakes connections between his/her own impressions for connections between things, in complex thinking the child groups objects based on their actual relations (Tuomi, 1998). Categorization is based on properties of objects,
although the groups do not reflect the relations between things in the same way as adult conceptual thinking. This type of thinking can therefore manifest itself as bizarre or idiosyncratic usage of mathematical signs. The complexes represent the strategies pupils use when attempting to assimilate culturally embedded concepts within both the school and everyday activities. At this stage the student is less egocentric, and therefore has access to both perceptual and actual conceptual bonds between objects. According to Berger (2005) Vygostky identified six types of complexes:

(a) Associative complex

(b) Chain complex: dynamic grouping in which each grouped object relates to the next in the chain, but there is no common abstract feature shared with all.

(c) Collection complex: grouping based on some trait in which the objects in the group differ

(d) Presentation complex

(e) Template orientation complex

(f) Pseudoconcept: The pseudoconcept is a construct that effectively bridges the gap between the individual and the social and between complexes and concepts. It is a special form of a complex that makes the transition from complexes to concepts possible. They resemble concepts, but the thinking behind them is still complex. Pseudo concepts occur whenever a student uses a particular mathematical object in a way that coincides with the use of a genuine concept, even though the student has not fully constructed that concept. Vygostky proposed that the use of pseudoconcepts enable communication between student and teachers which is essential for building new concepts, since at this stage the child correctly uses abstract concepts, but as a result of word meanings acquired from adults, not because the child’s
thinking would use spontaneously generated abstract concepts. These pseudoconcepts are generated as complexes in the child’s thinking but their word meaning coincides with concepts used by adults Tuomi (1998). Since it bridges conceptual thinking and complexes, this constitute the Zone of Proximal Development (ZPD), as thought processes change and advanced cognitive processes emerge.

Tuomi (1998) identified a sixth one called diffuse complex. Here groups of things that are similar not because of genuine likeness as judged by adults, but because the child thinks the objects have something in common.

According to Vygotsky, pure forms of thinking rarely exist even in adults. More advanced cognitive processes are built on earlier ones and fully formed conceptual thinking is only the most advanced form of thinking. Only the mastery of abstraction, combined with advanced complex thinking enables the child to progress to the formation of genuine concepts. Central to Vygotsky was the idea that complex thinking is based on socially shared and culturally inherited language. Words fulfil different roles in the various stages of complex thinking resulting in the following interaction;

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**Figure 2: A model of Vygotskian theory of the development of conceptual thinking: Adapted from Tuomi (1998)**
2.3.3 Thinking in concepts/ conceptual thinking

A mathematical idea becomes a concept (rather than a complex) when both its internal and external links are consistent and logical. Concepts are interrelated. According to Tuomi (1998), conceptual thinking requires more than unification of things. Some elements of the concrete experience need to be abstracted from the total experience. This means generalization and abstraction are equally important and undergo simultaneous development in the child’s thinking leading to conceptual thinking. Vygotsky proposed that there are two types of concepts, spontaneous and scientific, which are similar in the nature of their acquisition. Wellings (2003) argued that they both resist suggestion, have deep roots in child’s thinking, appear in more or less similar forms in children of the same age, have a long life in the children’s minds and die gradually unlike the suggested concepts and finally reveal themselves in the first correct answers of the child.

Vygotsky believes that spontaneous concepts are derived from children’s direct experience with world and their development is considered inductive. Since they are grounded in daily activities, formation is therefore through aggregation and synthesis of lived experiences. He further argues that the development of spontaneous concepts is dependent on pattern recognition, comparisons, reflection on activities and use of analogical reasoning. Vygotsky therefore considers them as strong in what involves
situational, empirical and practical cases. Wellings however points out that these concepts are however difficult to reason with, as forming abstractions in an inductive manner is a laborious and an inexact process.

On the other hand, the development of scientific concepts is considered deductive in which a child is exposed, within a structured school environment, to the abstracted conceptual knowledge of his or her culture. Wells(1994) cited in Wellings (2003) defines scientific concepts by comparing them to spontaneous concepts, when he says that while everyday concepts are related to the world of experience in a relatively ad hoc manner, scientific concepts are both abstract and general. Wells therefore argues that scientific concepts are characterized by generality, systematic organisation, conscious awareness and voluntary control. By generality and systematic organisation, he meant the characteristics of scientific concepts that structurally differentiate them from spontaneous concepts. On the other side by conscious awareness and voluntary control he referred to the manner in which scientific concepts are acquired and utilized. For Vygotsky, structured and systematic instruction in a school setting is important to the child’s acquisition of scientific concepts. This is what Tuomi (1998) meant when he asserted that while spontaneous concepts are created by the child based on empirical interactions with the object world, scientific concepts are acquired through social process of instruction.

Despite this distinction between these two types of concepts, Wellings (2003) pointed out that the two play an interdependent role in the development of super ordinate concepts. Thus he argues that while spontaneous concepts begin in the concrete phenomena of everyday life and scientific concepts begin with abstracted verbal definitions, one does not replace the other in concept formation. This therefore means that while the development of spontaneous concepts is required to facilitate the
absorption of scientific concepts, scientific concepts provide the system of meaning and the frameworks for new understanding that change the structure and organisation of spontaneous concepts. Vygotsky’s zone of proximal development (ZPD) determines how generative interaction between these concepts can occur. According to Vygotsky (1978), at the moment a child masters an operation, development is not completed, rather it would have started, which provides a basis for the development of a variety of difficult internal processes in children’s thinking.

2.4 Summary

It is this framework that forms a basis for analysing and categorisation of students’ mathematical thinking in this research. Students are therefore engaged in some form of thinking irrespective of the outcome. This thinking, according to Vygotsky can be categorized into heap, complex and thinking in concepts. Heap and complex thinking outcomes clearly constitute what teachers and other researchers have termed mathematical errors. The theory therefore is a good theoretical framework to help teachers analyze their students’ work and value their thinking in the classroom when learning mathematics.
Chapter 3: Literature Review

3.0 Introduction:

To gain insights on what has been done by other scholars on students’ mathematical thinking, this chapter presents a review of related literature. The first section of this chapter begins by looking at contemporary views on students mathematical thinking by other researchers. This is then followed by an outline of the purpose of analyzing students’ thinking in mathematics as well as a discussion of what other researchers have discovered about students mathematical thinking, and how these researches were carried out. The chapter concludes by highlighting the common forms of thinking, which other scholars have proposed to explain students thinking in mathematics.

3.1 Contemporary views on students’ mathematical thinking

Research findings on the use of children’s thinking in mathematics instruction suggest that developing an understanding of children’s mathematical thinking can be a productive basis for helping teachers to make the fundamental changes called for in current reform recommendations in mathematics education. Studies have been done in different forms, among them error analysis, review of other researches on students thinking and experimentally based researches in which designed programmes are instituted to see the benefits of allowing students to analyse their own thinking. Depending on the conclusion of such analysis, teachers should find corrective means and methods in order to deepen their students’ understanding of mathematical concepts, improve their reasoning methods and perfect their mathematical skills. This can only be achieved if teachers possess certain knowledge about students’ thinking and ways of responding to the students’ work. Analysing students’ thinking therefore aims to identify, expose and interpret thinking patterns for the purpose of modifying
instruction (Makonye, 2011; Nyahumwe, 2008; Borasi, 1994). As argued by Makonye this section of the study therefore focuses on other studies that have been done on analysis of students’ mathematical thinking.

Students errors in mathematics learning are a worldwide phenomenon and there is a long history of error analysis in mathematics education dating back as earlier than 1925 (Radatz, 1980). Most of the researches though have been an analysis of other researches done on this area (Radatz, 1980, Borasi, 1987, Batanero et al, n.d). According to Batanero errors and difficulties in mathematics do not arise in a random and unpredictable manner. Frequently, it is possible to discover regularities in them, to find some association with other mathematical concepts students would have constructed in the school or at home.

Peng (2010) pointed that analysing students’ errors is a fundamental aspect of teaching for mathematics teachers, a process which demands teachers to be knowledgeable in certain specific ways. Mathematical misunderstandings are now regarded as the key to greater understanding of how learners learn mathematics (Riccommi, 2005 cited in Makonye, 2011). According to Makonye, teachers should understand the nature of students’ understandings so that they can design appropriate strategies to help students understand mathematics better. This quest for understanding learners’ mathematical thinking in order to teach them more effectively falls under Pedagogical content knowledge. Shulman (1986) pointed that a teacher’s subject matter and/or pedagogical knowledge are not sufficient to help learners understand mathematics. He therefore argued that it is important for teachers to understand, among other factors, the common difficulties students encounter when learning a particular subject such as mathematics. Ma (1990) called this knowledge Profound Understanding of Fundamental Mathematics (PUFM). She described it as a
richly connected web of understanding that gave teachers a deep understanding of mathematics and ways to help students learn. It is imperative that teachers must be knowledgeable in this specific way as part of their pedagogical content knowledge. In the same vein Takker & Subramaniam (2012) argued that knowledge of students’ mathematical thinking includes knowing about students’ conceptions, their conceptual difficulties, potential learning trajectories and developing sensitivity to what students think and do in a mathematics classroom. Such knowledge supports opportunities for asking questions linked to the student’s ideas, eliciting multiple strategies for problem solving and drawing some connections across strategies (Franke, Kazemi & Battey, 2007 cited in Takker & Subramaniam). Fennema & Franke (1992) further assert that there is consensus that teachers’ knowledge is a major determinant of mathematics instruction and learning, an aspect that can be enhanced through analysis of students’ thinking. They however pointed that there is little agreement and even less evidence about what knowledge will enable teachers to teach students to learn mathematics with understanding. However an alternative basis for teachers to modify their instruction so that it is consistent with current recommendations is to change teachers’ knowledge (Clark & Peterson. 1986).

There is growing evidence that knowledge of children’s thinking is a powerful influence on teachers as they consider instructional change (Fennema & Franke, 1992; Putnam, Lampert and Peterson, 1990). Studies on maths education cited by Fennema et al (1996) reported that as teachers’ knowledge of students’ thinking grew, teachers’ knowledge of mathematics increased, their beliefs about mathematics and instruction were modified and instructional change occurred. Instructional decisions would therefore involve; the choice of problems for children to solve, questions to ask to elicit
students’ understanding and thinking as well as ways to assist children in learning to solve problems and report their thinking.

Fennema et al (1996) proposed that one way to find if students have understood is to ask them to explain their thinking. In studies where students were asked to share their thinking and their thinking valued, there were gains in students achievement (Fennema et al, 1996; Borasi & Fonzi, 2002). Rhine (1998) cited in Borasi & Fonzi suggests that it is important to foster an attitude that values analyzing students’ thinking as part of teachers’ everyday practice and provides strategies to help them do so. In addition Darling-Hammond (1998) (ibd) sums it well by emphasizing that teachers need to be able to analyze and reflect on their practice to assess the effects of their teaching and to refine and improve their instruction. It follows then that there is overwhelming evidence to support the importance of analyzing students thinking.

Despite such an important role played by knowledge of students’ thinking and understanding, it has been discovered that teachers are usually afraid to investigate their students’ thinking, in particular thinking that results in revealing mathematical errors and misconceptions. According to Yackel et al (1990) many teachers not only fail to see the value in the feedback of students’ misunderstandings, teachers are somehow threatened or irritated by the misunderstandings. In the same vein Legutko (2008) advised that teachers should not be afraid of students’ misunderstandings, but should create such situations in which students reveal their thinking so that teachers are able to methodologically direct and correct them. She highlighted that mathematics creates an internally coherent structure and some concepts are built on the basis of other concepts, therefore learning the subject is difficult. Similarly Makonye (2011) argued that mathematical knowledge cannot be transmitted from teacher to students without their own way of understanding, rather knowledge must be dynamically re-
interpreted, reorganised and reconstructed in each learner’s mind. When this happens some of the understandings that the learners construct are inconsistent with the expected, and therefore can hinder progress in learning mathematics. He further explained that learning is hindered because mathematics is a hierarchical subject, as learning at higher levels is dependent on appropriately attained requisite knowledge and skills at lower levels.

A seemingly small gap in comprehension or knowledge creates further misapprehension that are built one upon another and which after sometime is revealed in an error avalanche. Legutko (2008) further concludes that an unrevealed error, which is rooted in the mind of the student, is therefore a major threat to the construction of mathematical knowledge. In her paper ‘An analysis of students’ mathematical errors in the teaching-research process’ she suggested that various contradictions are rooted in mathematics itself resulting from among them; attempts at algorithmization versus creative and conscious actions, natural thinking of everyday life versus formal reasoning based on accepted conventions and abstraction of mathematics as a science versus connections of mathematics with the real world.

These and other contradictions create misunderstandings and might be the reason for numerous students’ errors (Krygowska, 1988a, Rouche, 1988 & Pellerey, 1988 cited in Legutko). Understanding these errors and misconception will therefore help teachers understand their students thinking as they learn mathematics. On the other hand Harel & Sowder (2005) proposed that thinking has been categorized as flawed thinking (relying solely on empirical observations to justify mathematical arguments or over generalising mathematical ideas (Matz, 1980)) and sound thinking (looking for elegant solutions to problems and generalising mathematical ideas). Similarly, Brousseau (1997) cited in Harel & Sowder characterized obstacles in developing ways of thinking
as didactical obstacles and epistemological obstacles. He proposed that didactical obstacles are a result of narrow or faulty instruction while epistemological obstacles are manifested in mathematical knowledge and therefore unavoidable due to the nature of development of this knowledge. As a follow up to this distinction, Duroux (1982) in Brousseau considers conditions for a piece of knowledge to be considered an epistemological obstacle as having traces in history of mathematics and that it is not a missing conception or lack of knowledge, rather it is a piece of knowledge or conception that produces responses that are valid within a particular context and generate invalid responses outside this context. It is these responses generated out of context that are termed errors. The level of acquisition of a way of thinking by an individual is therefore determined by the extent to which the individual has overcome these obstacles.

More often teachers seem too busy to pay attention to their students’ errors. Students continue to make errors and teachers sometimes become frustrated, blame students for being dull and seem not to bother to analyse them to inform their teaching methodology. It is therefore hoped that this research will make a small contribution to the growing calls to make analysis of students’ thinking part of the teachers’ pedagogical content knowledge, thereby making students’ voices heard in teaching mathematics. It is however important to note that the challenging issue concerning students’ misunderstandings is that many students have difficulty in relinquishing misconceptions because the false concepts may be deeply engraved in their mental map (Muzangwa & Chifamba, 2012). Tobey & Minton (2010) concurred by acknowledging that misunderstandings interfere with learning when students use them to interpret new experiences and that students are emotionally and intellectually attached to misunderstandings which they can only give up with great reluctance. This
observation is aptly summed by Hammer (1996) cited in Muzangwa & Chifamba that misunderstandings, which are strongly held stable cognitive structures, affect in a fundamental way how students understand natural phenomena and scientific explanations and must be overcome, avoided or eliminated for students to achieve expert understanding. Teachers must therefore thoroughly prepare themselves to handle misunderstandings and help students give up these misunderstandings or avoid them.

It follows that teachers can only get prepared to handle students’ misconceptions with confidence if they understand the process that gives rise to these, which is students’ thinking. Yackel et al (1990) summed it well when they argued that when a child gives an incorrect answer, it is especially important for the teacher to assume that the child was engaged in meaningful activity. Thus a teacher should help child reflect on his or her solution attempt to evaluate it. The promotion of views raised in this section, that promote effective mathematical understanding requires teachers to assume that each student’s mathematical understandings are personally meaningful and take responsibility to help students verbalise these understandings in a mathematically meaningful way. There is consensus among researchers that the analysis of students’ thinking is a resource that helps teachers in their pedagogical decisions and consequently helps improve their teaching. ((Makonye, 2011; Doerr, 2004; Kazemi & Franke 2004; Steinberg, Empson and Carpenter, 2004 cited in Peng & Song, 2008; Yackel et al, 1990).
3.2 Purpose of analysing students’ thinking

Research contributions to analysis of students thinking is essentially concerned with the opportunities analysis offers to the mathematical teacher, on curricular and methodological activities, and with the tests of the efficiency of special training programs aimed at reducing error frequency. Radatz (1980) listed five early research endeavour objectives of error analysis which are; listing all potential mathematical errors, determining the frequency distribution of these error across age groups, analysing special difficulties particularly encountered when doing written division and operating with zero, determining the persistence of individual errors and attempting to classify and group errors.

Over the years error analysis research has developed beyond the general aim of diagnosing errors for remediation. It now includes efforts to sensitize teachers for the so called “diagnostic teaching of mathematics” in which analysing students’ thinking play an important role in their learning of mathematics (Ashlock, 1973, Reisman, 1972, Robitaille, 1976 cited in Radatz, 1980) or informs them of this method. Some researches justified analysis of students’ thinking in the form of errors by just looking at its role in learning mathematics. Borasi (1987) for example, considered error analysis as a diagnostic tool and as a springboard for further inquiry for both the teacher and the student. According to Radatz student errors illustrate individual difficulties. Analysing them may reveal the faulty problem solving process and provide information on the understanding of and the attitudes toward mathematical problems. From his perspective the importance of error analysis is twofold:

1. With regard to the requirements of academic practise, as an opportunity to diagnose learning difficulties, as a method of developing criteria for
differentiating mathematics education and as a means to create more awareness and support for the performance and understanding of individual students.

2. Error analysis seems to be a remarkable starting point for research on the mathematics teaching-research process. It must be considered a promising strategy for clarifying some fundamental questions of mathematics learning.

(plt: 16)

Various causes of errors that cut across mathematical content can be identified by examining the mechanisms used in obtaining, processing, retaining and reproducing the information contained in mathematics tasks (Radatz, 1979). It is therefore imperative that consideration of the diagnostic and causal aspects of errors could give specific help to mathematics teachers by allowing them to integrate their knowledge of curriculum content with their knowledge of individual differences in children thus making their pedagogical content richer. The attention given to students’ voices in the classroom is hoped to make teachers better teachers who teach mathematics for understanding. This research will help provide support for teachers by characterizing errors using Vygotsky’s theory of concept formation, an exercise that is hoped to make teachers understand students’ thinking better. The purpose of error analysis for the current study could therefore be summed as; firstly, to describe students’ mathematical thinking when learning differentiation at advanced level, secondly, to understand why students think in ways that are considered unusual when learning differentiation at advanced level and thirdly to categorize students’ mathematical thinking when learning differentiation at advanced level.

Finally, it cannot be overemphasised that students’ thinking analysis is a critical component of a teacher’s work as it provides insights on pupils’ challenges, their
misconceptions and their sound understanding, and consequently informs the teacher on how to help them better, or how to use their sound understanding to clear their misconceptions and learn other mathematical concepts.

3.3 Research findings on analysis of students’ mathematical thinking

Research has been done on analysis of students’ thinking using varying methodologies. The focus however has been on analysing and categorizing errors rather than analysing and categorizing the thinking process resulting into these errors. Peng (2010) focused on teachers’ knowledge of students’ mathematical errors rather than the errors as a product of thinking. The results of her study indicated that although teachers’ knowledge of students’ errors differs in different tasks, there are emerging patterns on the extent of how knowledgeable mathematics teachers are about students’ mathematical errors. Brousseau (1981) cited in Peng (2010), who used historical elements in order to explain pupils’ errors in decimal fractions, found that pupils make the same errors independently of the teaching methods used, and therefore concluded that there are errors that can be attributed to pupils’ epistemological foundations. Vosniadow and Verschaffel (2004) cited in Peng (2010) asserted that there is the widely recognized conceptual change framework, within which errors initially conceptualized negatively are now seen as a natural state in knowledge construction and thus inevitable.

Peng and Luo (2009) developed their own theoretical framework to investigate mathematical teachers’ knowledge as used in analysing students’ thinking as manifested in the errors they make. The framework includes two dimensions; nature of mathematical error and the phases of error analysis, which are closely linked in a complex way. Based on this framework the research identified four levels of teachers:
Level 1 could not identify students’ errors, level 2 could identify errors but could not find underlying reasons, level 3 could identify errors and find reasons and level 4 could identify and find reasons in a right and quick way. Although the study focused on teachers’ knowledge of errors more than analysing these errors in the context of concept formation, it nevertheless showed the importance of analysing students’ mathematical thinking as contributing to pedagogical content knowledge. Peng and Song (2008) in their study, inquiry into the students’ mathematical thinking through error analysis as a means to teacher development, had earlier on concluded that understanding students’ mathematical understandings require teacher to understand students’ thinking patterns. In their words they asserted that if we understand why the right is right deeply, then we can understand deeply why the wrong is wrong and vice-versa.

Legutko’s (2008) analysis of students’ errors in teaching – research process studied teachers’ responses to students’ errors and the teachers’ strategies to deal with the errors. The study recommended that teachers should understand students’ misunderstandings, contemplate their causes and methodologically correct them. It was however noted that, teachers are generally afraid of student errors. This fear is often manifested by teachers asking students too many questions in order to navigate them to a correct answer and avoid the error. A revealed error requires explanation and its correction takes time. It is therefore useful for teachers to speculate about possible student errors while preparing their lessons (predicted error). The fear of students’ misunderstandings was also seen to come from the realisation that these misunderstandings might reflect badly on their teaching. Students’ mathematical thinking that results in errors is often seen as a revelation of lack of skill or comprehension which should have been mastered earlier on in the learning process.
Frendenthal (1989) (as cited in Legutko, 2008) concurred that students who make errors always do so with the teacher who teaches them, at least partially: the error’s role is connected with the teacher’s role in the learning process.

On the contrary individuals’ idiosyncratic construction of meaning, (systematic error patterns) have been attributed to constructivism by other researchers rather than being taught. Misunderstandings are taken as evidence for the constructive nature of knowledge acquisition (Confrey & Kazak, 2006; Ernest, 1995; Nether, 1987 all cited in Makonye, 2011). In their strategy to deal with misunderstandings, teachers are advised to accept students’ right to err, especially when students face a new unusual situation. A familiar action scheme cannot be immediately applied in an unusual situation. It has to be either adjusted or a complete new scheme has to be formed to solve the problem. Booker (1989;99) cited in Legutko (2008) added his voice by advising teachers to try to understand students’ errors, try to understand the way students think because ‘children do not make errors in mathematics thoughtlessly; they either believe that what they are doing is correct or are not at all sure what they are doing. It is therefore imperative that to understand their thinking process, teachers must understand errors children make as well. The study therefore concluded by suggesting to methodologically correcting errors by; trying to make students aware of the errors by questioning, if questions are not helpful, create contradictions, contrasts or counter examples if students do not correct themselves, teachers can use other students’ help and lastly sometimes it is possible (or even necessary) to postpone the error discussion for the next class.

Some studies gave sources and reasons for mathematical errors to sensitize teachers on how to manage students’ errors. Batanero et al (n.d.) singled out cognitive
obstacles as the main source of students’ errors. Brousseau (1983) cited in Batanero et al identified some features and kinds of these cognitive obstacles;

(a) An obstacle is knowledge, not lack of it
(b) Students employ knowledge to produce the correct answer in a given context
(c) When this knowledge is used outside this context it generates mistakes
(d) The student ignores contradictions produced by the obstacle and replaces it in the new learning
(e) After the student has overcome the obstacle, recognising its in exactitude, nevertheless, it recurs sporadically.

The kinds of obstacles specified as a result of these features are ontogenic obstacles due to features of child development, didactical obstacles from didactical options chosen in teaching and epistemological obstacles which are intrinsically related to the concept itself, and carry part of the meaning of the concept. This classification is an acknowledgement that students do have misunderstandings and as such, errors are not spontaneous, but a result of their experiences in the teaching-learning process. Therefore, to enhance concept formation, teachers must understand the nature of students’ thinking. Another method that has been used to analyse students’ thinking (in the form of errors) is to classify them into certain categorizations based on an analysis of students’ behaviour. Radatz (1979) based his analysis on an information processing model and therefore identified reasons for errors as

(a) Errors due to language difficulties. Maths is like a foreign language for students who need to know and understand mathematical concepts, symbols and vocabulary.
(b) Errors due to difficulty in obtaining spatial information
(c) Errors due to deficient mastery of prerequisite skills, facts or concepts
(d) Errors due to application of irrelevant rules or strategies
(e) Errors due to incorrect associations or rigidity of thinking.

Since errors show a way of understanding, and therefore an indicator of how students think, Radatz (1979) confirmed the widely supported view that errors in learning mathematics are not simply the absence of correct answers or the result of unfortunate accidents. They are the result of definite processes whose nature must be discovered (Ginsburg, 1977, Menchinskaya & Moro, 1975 cited in Radatz, 1979). Exploration surveys have also made important contributions to the need for error analysis (Borasi, 1987 and Radatz, 1980). While analysing exploration versus diagnosis Borasi acknowledged that the extensive body of research on error analysis in mathematical education has already made us aware of the value of probing a mistake such as \( \frac{1}{4} + \frac{6}{7} = \frac{7}{11} \). Such a mistake may contain valuable information about specific difficulties that the student encountered in learning fractions and about his/her conception of fractions, rules and mathematics. Borasi showed the unlimited probing opportunities that can be raised from such an error. His analysis showed that what is happening here is clear but reasons may not be obvious. Possibilities were shown to include just confusing rule for adding fractions with rule for multiplying fractions, trying to operate with fractions as she/he did when adding whole numbers and that this way of operating may seem appropriate in real life situations such as baseball betting or keeping of records of game results. The analysis therefore becomes a valuable tool to understand the students thinking and provides a basis to help them effectively. The reasons students make errors can be therefore summed as an attempt to associate new situations to what they already know in their everyday life or what they have
experienced in class. Remediation is only possible if teacher is willing to look at these possible causes and relate them to individuals rather than just write comments or condemn students as slow and dull.

Borasi (1987) further suggested interpreting errors as springboards for exploration. The study noted that interpreting errors solely as tools for diagnosis and remediation would have only partially exploited the educational potential of errors. The focus of diagnostic studies is on avoiding the errors and not actually on utilizing the errors didactically as showing how students would have thought to arrive at the answers (Verwey, 2010). Within such a context the students would be deprived of the opportunity of engaging in the activity of attempting to ‘explain’ and ‘fix up’ their own errors, something that would prove motivating and challenging. Using students thinking as motivation and means of inquiry into the nature of mathematics could improve students understanding of mathematics as a discipline and improve students’ achievement in mathematics (Borasi & Fonzi, 2002 and Fennema et al, 1996).

The conclusion by Fennema et al was a result of their research using Cognitive Guided Instruction (CGI). CGI is a project designed to enable teachers understand their students thinking (Carpenter & Fennema. 1992). It is an elementary level mathematics professional development programme developed at Wisconsin Centre for Educational Research in the 1980s and 1990s by professors Fennema, Carpenter and their colleagues Levi, Empson and Jacobs. It is a research programme that involves presenting problems to students and then teacher asks them to think about ways to solve the problem. The major goal of CGI is to help teachers develop an understanding of their own students’ mathematical thinking and how their thinking could form the basis for development of more advanced mathematical ideas. It is a research based model involving analysing a number of generated responses from
students’ solutions to a given problem, in particular being built around identified irregularities in children’s solutions.

Borasi (1987) therefore offered two directions along which errors can be used to motivate reflection and inquiry about nature of mathematics that are;

1. Errors can be used to investigate the nature of fundamental mathematical notions such as proofs, algorithm or definition since these are continually used when studying mathematics and

2. An analysis of the variety of ‘degree of wrongness’ among mathematical errors can help clarify the nature of ‘truth’ in mathematics errors.

These two help to investigate abstract issues regarding nature of mathematics, and therefore inform teachers about students thinking when learning mathematics.

From the literatures it can be observed that there are insights from studies in both analysing students’ mathematical errors and mathematics teacher knowledge as well as how the teachers are knowledgeable of students’ mathematical errors. Common aspects explored by the studies include error analysis as a diagnostic tool, the need to identify and correct errors and errors as a source of motivation for further exploration. There is general agreement among studies that teachers must understand students’ thinking to help them structure their teaching in ways that help students learn better and avoid perceived errors. Although studies seem to be about errors and misconceptions, rather than students’ thinking, these are relevant to the current study.

The main reason for errors is the misconception of over generalising. As noted by Davydov (1995) in Makonye (2011), generalisation is an inseparable rule linked to the process of abstracting. Makonye noted that it involves deliberately extrapolating the range of reasoning beyond the specific examples considered attempting to expose
commonality across situations. Thus when students make errors, they will be thinking, attempting to generalise across areas. He summed it all when he argued that if students’ flawed mathematical reasoning and thinking is not deconstructed, unpacked and unveiled to both teachers and learners for reconsideration and reflection then little progress will be made in learning since teaching will continue to be misdirected away from the challenges facing learners. Teaching must therefore connect with learners’ needs in the form of misunderstandings to be relevant. There, however, seem to be no research encountered that analyse the thinking patterns of students to understand how misunderstandings come about in concept formation.

Although some studies used other different frameworks to explain errors, they seem to perceive errors negatively rather than as a way of thinking in concept formation as advocated by Vygostky’s theory of concept formation. Verwey (2010) cited two studies done by Henze (2005) and Santagata (2005) that reported rather disturbing evidence that correcting of errors is not the main focus of the classroom events with teachers ignoring a considerable number of learner errors during classroom discourse. In cases where errors were attended to, the focus was on getting correct answers rather than analysing the wrong answer to understand students’ thinking patterns. This research aims to fill this gap.

Studies done on differential calculus have highlighted some thinking patterns by mathematical students, although there has been little characterization done to explain why students think in that way. Although studies identified by Makonye acknowledged that some students can successfully differentiate, they often fail to account for the concepts that underpin the techniques of differentiation apparently applied. The topic of differentiation was noted to cause many errors because it is associated with many other concepts. Some errors peculiar to differentiation include problems with notations
\( \frac{dy}{dx} \) or \( \frac{dy}{dx} \) as well as \( dx \) or \( dy \) which are not really meaningful except when used together as \( \frac{dy}{dx} \) or with integration, \( \int f(x)dx \) (Makonye, 2011). Students in Orton’s (1983a) study cited in Makonye on errors on differentiation produced many unsatisfactory responses which appeared to suggest widespread misconceptions.

Vinner and Amit (1990) (as cited in Makonye) reported errors and misconceptions in executing procedures for optimisation problems. Optimisation problems were reported to give students challenges due to language barrier. Orton reported that students found it difficult to understand the rate of change of gradient at a point as a limit yet according to Golden (n.d) one of the central questions that differentiation seeks to answer is what is meant by the rate of change of a function \( f \) locally at a point \( x \), is \( \frac{dy}{dx} \) a fraction or a single indivisible symbol, can \( du \) be cancelled in \( \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \)? On categorization of thinking in differentiation, Makonye (2011) analyzed students’ thinking (in the form of errors) by categorizing them into predetermined categories, among them:

- **Executive errors** in which learner has knowledge about mathematical concepts but fails in carrying out processes such as simplification,
- **Semantic errors** in which the student resort to prior understanding for dealing with concepts in totally different contexts. This can be shown in multiplication of algebraic terms versus the function notation such as \( f(x + h) = fx + fh \).
- **Structural errors** in which students use concept image that is not adequate. Students' thinking shows failure to grasp the principles essential to the problem and therefore termed them conceptual errors.
• Hybridisation of errors in which students attempt to apply ideas where they do not apply. Makonye gave an example of associating $f(x)$ with transformations and pointed that students fail to do calculus because of algebraic executive errors.

• Pseudo linearity which included $(a + b)^2 = a^2 + b^2$, $\frac{d}{dx}(f(x) \times g(x)) = f^1(x) \times g^1(x)$ and $\log(x - y) = \log x - \log y$.

From the literature it can be noted that the bulky of research done on error analysis was at elementary and university levels, with very few researches done at advanced level. Although some of the errors are from algebra and trigonometry, such errors have been seen to be very common when learning the concepts of differentiation (Makonye, 2011). Having taught ‘A’ level mathematics for fifteen years I have observed that teachers rarely give themselves time to listen to their students (an observation supported by Santagata, 2005; Henze, 2005 cited in Verwey, 2010), something that I also do myself. Teachers are too busy teaching for students to pass examinations, without creating opportunities in their classroom for students to be heard and their thinking appreciated and valued. More often than not students pass examinations without understanding. This research will therefore, though in a small way, help teachers understand their students’ way of thinking and come up with tailored instruction which is hoped will improve mathematical understanding.

3.4 Thinking Forms in Mathematics

According to Henningsen and Stein (1997) thinking processes can range from memorization to the use of procedure and algorithms (with or without attention to concepts, understanding or meaning) to complex thinking and reasoning strategies that would be typical of “doing maths” (conjecturing, justifying or interpreting). Different
scholars have attempted to define mathematical thinking by looking at ways of thinking or thinking styles. Some have tried to explain thinking mathematically by explaining ways of understanding as these influences students’ ways of thinking. Harel & Sowder (2005) proposed that the particular meaning students give to a term, text or sentence, the solution they give to a problem or justification they use to validate or refute an assertion are ways of understanding, while the general theories, implicit or implied underlying such actions are ways of thinking. On the other hand Ferri (2012) used mathematical thinking styles (referred to as Theory of Mathematical thinking Styles, MTS) to explain how students learn mathematics. She therefore believes that as they learn mathematics, students use visual, analytic and integrated thinking.

3.4.1 Visual Thinking

Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations, as well as preferences for understanding of mathematical facts and connections through holistic representations. The internal imaginations are mainly affected by strong associations with experienced situations. Visualisation is the ability to process, interpret or reflect upon pictures, images or diagrams in the mind, on paper or with technological tools, with the purpose of developing previously unknown ideas to advance mathematical understanding (Zimmermann & Cunningham, 1991; Hershkowitz et al 1989 and Arcavi, 2003 cited in Rosken & Polka, 2006). Although visualisation is thought to be a powerful way to explore mathematical concepts and that it reduces complexity, Arcavi believes that it is considered cognitively more demanding than analytical techniques as it rely on techniques that are not procedurally safe. Tall and Vinner (1981) viewed this thinking style in the context of concept images which they think includes visual images,
properties and experiences concerning a particular mathematical concept. To understand a concept requires that the student develops a particular image of it.

3.4.2 **Analytical thinking**

Analytical thinkers show preference for internal formal imaginations and eternalized formal representations. They are able to comprehend mathematical ideas through existing symbolic or verbal representations and prefer to solve mathematical problems in a sequence of steps (Ferri, 2012).

3.4.3 **Integrated thinking**

This way of thinking combine visual thinking and analytic thinking and integrated thinkers are able to switch flexibly between different representations or ways of thinking.

3.4.4 **Analogical Thinking**

According to Gick & Holyoak (1983) analogical thinking involves the transfer of knowledge from one situation to another by the process of mapping. Reasoning by analogy thinking implies a comparison of two concepts (analogs) at the same (usually quite concrete) level of abstraction. In analogical problem solving, one problem and its solution are already known and the analogist simply notes correspondence between the known problem to a new unsolved one and on that basis derives analogy.

3.4.5 **Intuitive thinking**

Bruner (1960) believes that intuition implies the act of grasping the meaning, significance or structure of a problem or situation without explicit reliance on the analytic application of one’s craft. The rightness or wrongness of intuition is finally
decided not by intuition itself but by the usual methods of proof. An intuitive thinker therefore suddenly comes with a solution for which he has to provide a proof through experience. Intuitively one quickly makes good guesses about something or about which among several approaches to a problem will prove fruitful.

3.4.6 **L-modal and R-modal thinking**

Wachsmuch (1981) believes all the different forms of thinking can only be categorized into two groups in which, in one, logical thinking and language seem to be always involved whereas in the other, standards of thinking are somewhat relaxed. These two groups are termed L-mode and R-mode thinking. L- Modal thinking means concentration and the conscious sequencing of trains of thought which otherwise would appear concurrently. On the other hand R-modal thinking means relaxation and leaving out detail as it favours parallel, holistic thoughts of broader range which would sometimes yield spontaneous insights. These thinking modes not only involve imagery and logic when solving problems, they also involve a non casual unconstrained liberal way of thinking which may yield illogical mental links. These modes do not operate in isolation; hence their cooperation could involve the interplay of creative and productive thinking important in mathematics.

3.4.7 **Dual Process Theory (DPT): intuitive and analytical thinking.**

Leron and Hazzan (2006) offers another theory that classifies thinking in mathematics into intuitive and analytical thinking. These modes mostly work together to yield useful and adaptive behaviour and sometimes giving non-normative answers. According to the theory our cognition and behaviour is driven by parallel and different modes called System 1(S1) and System 2(S2) roughly corresponding to our common sense notions of intuitive and analytical thinking. S1 processes are characterised as fast, automatic,
effortless, unconscious and inflexible while S2 are slow, conscious effortful and relatively flexible.

The two differ in accessibility. In most situations S1 and S2 work together but sometimes S1 may generate quick automatic non-normative responses while S2 may or may not intervene in its role as monitor or critic to correct or override S1. Many non-normative answers given in mathematics tasks can be explained by the response of S1 and the frequent failure of S2 to intervene. Stanovich (2004) cited in Ejesbo & Leron (in press) argues that in these cases, the effortful S2 is not alerted and students accept S1’s output uncritically and thus behave irrationally. In some cases, although S1 jumps with an answer, in the next stage S2 interferes critically and the students then make necessary adjustments to give the correct answer. Evans (2009) in Ejesbo & Leron offers a different view, which he termed Default interventionist Approach of the relationship between S1 and S2. He proposes that S1 has access to all incoming data and its role is to filter it and submit to its suggestions for S2’s scrutiny, analysis and final decision. This is plausibly an efficient way to act for S2, but it is error prone because the features S1 selects are the most accessible, but not always the most essential.

3.5 **Summary**

This chapter reviewed related literature. This was done by looking at the contemporary views on students thinking in mathematics, purpose of analysing students’ thinking in mathematics, other researches done on students’ thinking and forms of thinking in mathematics. This helped to focus the current research and get insights on what other researchers say about students’ thinking. The chapter offered other forms of categorizing students’ thinking, which was mainly in the form of errors categorization.
While most of the reports cited in this chapter acknowledge the importance of analysing students' thinking, none among those cited provided an in depth analysis of students' thinking using a theoretical framework such as Vygotsky' theory of concept formation.
Chapter 4: Research Methodology

4.0 Introduction

In this chapter, the research methodology is outlined. The research design and paradigm are given as well as justification for the chosen research design. The purpose of the current study is to investigate students’ mathematical thinking when learning differentiation. This study is qualitative with the case study research design used. In qualitative researches, the researcher is the primary instrument for data collection (Guba & Lincoln, 1985; Bogdan & Birklen, 1998). The researcher needs to be at the centre of data collection activities to subjectively select events to investigate (Nyawaranda, 1998) and so is the primary instrument for data collection and analysis (Makonye, 2011). Qualitative study methodology facilitated this research as it concerns description of thinking processes exhibited in learners’ answers. According to Makonye (2011), qualitative research can be described as interpretive or constructivist with emphasis on understanding the world through the perceptions of its participants. The research techniques that were mainly used are document analysis, observation and task based interviews. Data was collected over a three week period of teaching the topic of differentiation.

4.1 Research design

Case study research design was used in this study. The aim of this research is to explore students’ thinking when learning differentiation at ‘A’ level. Since this research was context specific and the role of researcher being one of inclusion, with the objective of understanding students ‘thinking patterns a qualitative paradigm was found to be suitable.(Cohen, Manion & Morrison, 2000). Qualitative methodology was appropriate because the research into understanding thinking processes required a
detailed description of what students think in order to capture the essence of what they would have written in response to maths basic domains. It also seeks patterns and relationships as well as descriptions of thinking processes as exhibited in learners’ responses (written or verbal), making qualitative methodology appropriate. According to Merriam(1992) in Makonye(2011) achieving a deep understanding in a specific phenomena, to probe beneath the surface of a situation and to provide a rich context for understanding the phenomena under study is the aim of qualitative research. In other words, it aims to uncover prevalent trends in thought and opinion. Qualitative research can be described as ‘interpretive’ or ‘constructivist’ with emphasis on understanding the world through perceptions of its participants (Bryman, 2004; Hatch, 2003; Yin, 1994 cited in Makonye).

4.2 Data collection procedure

A case study of Monte Cassino ‘A’ level students was conducted with researcher being the only teacher participating in the analysis of students’ thinking. Since this study is action research, data was collected during the teaching of differentiation mainly through document analysis and task based interviews. The researcher identified thinking patterns as he went through exercise books while marking the students’ work. Interviews were done on selected pupils’ work whenever teacher discussed students’ work with individuals to get insights on their way of thinking. Finally, observation was used each time class discussions were conducted and errors noted whenever pupils were asked to present solutions to questions on the chalkboard.

Since the researcher is by profession a Maths teacher at secondary school level, it was natural to go into the classroom as a participant observer. This opened
opportunities for interaction with the students to understand their thinking. What the students expressed in action, words and writing therefore constituted empirical data for this study. Written observation protocols were kept stating what was noted and perceived as forms of thinking during the lesson.

4.3 Instruments

4.3.1 Task Based interviews

Task-based interviews were used to collect qualitative data. The study aimed to analyse how students think when learning differentiation. Questions were designed to elicit responses from the pupils in written form. After analysing the written work, selected pupils were interviewed to gain insights on their thinking patterns.

According to the National Council of Teachers of Mathematics, mathematical tasks are central to students’ learning because “tasks convey messages about what mathematics is and what doing mathematics entails”. (NCTM: 1991:24). The tasks in which students engage provide the contexts in which they learn to think about subject matter and different tasks may place different cognitive demands in students (Doyle, 1983, Marx & Walsh, 1988, Heibert & Wearne, 1993 cited in Henningsen & Stein, 1997). Thus the nature of tasks can potentially influence and structure the way students think and can serve to limit or broaden their views of the subject matter which they are engaged. Data was therefore generally collected by analyzing students’ written work, carrying out clinical interviews as a follow up to tasks given as well as discussion of cases or narratives of classroom experiences created to highlight the mathematical thinking and activities of selected students.
4.3.2 Interviews

The interviews were done to probe thinking patterns of students. Individual students were purposively selected by the teacher for the interviews as a follow up to their responses to given tasks either as written work or as discussion questions. Some interviews were done during lessons as teacher probed pupils when asked to make presentations on the chalkboard. The interviews were recorded by using a voice recorder which was then transcribed for presentation. The following questions were used as a guide:

1. Can you explain what you did?
2. How do you know?
3. What makes you think so?
4. Can you say more about your method?
5. Is there an alternative way to arrive at the answer?

The sequence of the questions and the actual questions asked depended on the student’s response at any given time.

4.3.3 Observation

Observation was also used whenever students were asked to discuss tasks that would have been given as pair work or class discussion. Whenever pupils were given cases or problems for discussion, the teacher observed and captured the different reasoning patterns demonstrated by the pupils as they exchanged ideas. Discussion activities were in the form of tasks designed by the teacher to capture students thinking or contexts taken from other students’ solutions (artefacts).
4.3.4 **Document Analysis**

The research involved analysing pupils' work so what they expressed in written form constituted the bulky of the data. Their exercise books were used as a source of contexts to be analysed by the teacher and class discussions.

4.4 **Sample**

A naturally assembled group (class) with 35 students was used in this study. A school class was used since it is located close enough and therefore allowed frequent participation of the researcher and to ensure a socially stable area with little movement in and out of the school. In a qualitative study, the sample is usually purposively selected. It is usually a small non-probability sample that is non-representative (Makonye, 2011). The non-representativeness is not important because findings of the study are not generalisable to the population, but are important in generating theory that may be later tested in a general population (Yin, 1994 in Makonye).

4.5 **Data analysis**

Data were analysed qualitatively. Data analysis began soon after data collection started as the researcher checks on research questions and anticipated results. Content analysis technique was used to analyse the collected data. Makonye (2011) pointed that content analysis is an instrument of data analysis used in social sciences that involves analyzing written texts which is categorized and classified. According to Holsti (1999) cited in Makonye, it is a technique for making inferences by objectively and systematically identifying specified characteristics of messages, which in the current research involves the written and verbal responses. In this study I analysed trends in thinking and relate these trends to known characteristics (according to
Vygotsky’s theory) of learners to their thinking. The different levels of thinking identified during lessons, and in pupils' work were analysed and classified using Vygotsky’s theory of concept formation.

4.6 **Summary**

This chapter outlined the methodology used in the study, the research design, sampling procedures as well as data collection instruments. The research instruments informed the research on how to answer the research questions on students’ thinking in learning differentiation. Content analysis was used to analyse the data collected, which was categorized using Vygotsky’s theory of concept formation.
CHAPTER 5: Data presentation, analysis and interpretation

5.0 Introduction

In this chapter data is presented by grouping it into: Differentiation from first principles and use of the rule \( \frac{d}{dx}(x^n) = nx^{n-1} \), differentiation of products and quotients, differentiation of exponential and logarithmic functions, implicit differentiation, parametric differentiation and optimisation problems. Based on the students’ work both written and verbal thinking portrayed by the students is categorized in this chapter into heap, complex thinking and thinking in concepts. The results of the current research will be compared to what other researchers have discovered about students thinking when learning differentiation as mentioned in the review of related literature.

5.1 Differentiation from first principles and use of \( \frac{d}{dx}(x^n) = nx^{n-1} \)

After the introduction of differentiation and deriving the rule for differentiation from first principles, pupils were asked to use the result to differentiate \( y = x^2 \), to establish the result \( \frac{d}{dx}(x^n) = nx^{n-1} \) through the use of a pattern during a lesson. One student had the result \( \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} \)

\[
= \frac{x^2 + \delta x^2 - x^2}{\delta x}
\]

\[
= \frac{\delta x^2}{\delta x}
\]

\[
= \delta x.
\]

When asked to further explain her answer, the student explained that since
\( y = x^2, \) then \( f(x + \delta x) = x^2 + \delta x^2. \) The teacher tried to lead the student to identify the errors inherent in her work. When asked to write \( f(x) \), she wrote \( y = x^2. \) The teacher reminded her to use the \( f(x) \) notation and she then wrote it correctly. When further asked to write \( f(x + \delta x) \) below her \( f(x) \) so that she gets a hint on the substitution she failed to do this. It seemed the failure was a result of language barrier as there was communication breakdown. This observation is consistent with Radatz (1979), when he reported that some students’ errors are a result of language difficulties as Mathematics is like a foreign language to them. When the discussion was opened to the class, some students managed to identify that there was use of an incorrect notation \( f x \) instead of \( f(x) \), omission of \( \lim_{\delta x \to 0} \) in the subsequent lines in the working despite it being present on the first line. Another student then presented the following as a follow up to what the teacher had advised the other one to do:

\[
f(x) = x^2
f(x + \delta x) = x^2 + 2x\delta x + (\delta x)^2,
\]

then used these corrections to obtain a correct result. The first student displayed both heap and complex thinking. The heap thinking manifested itself when she wrote \( f x \) for \( f(x) \). This seemed to emanate from algebra when brackets are used for multiplication and she believed its true for all algebraic terms that \( f(x) = f x = f \times x. \) The omission of the limit is consistent with what other researchers on calculus have reported, since in most problems in which students used differentiation from first principles the students either omitted the limit or failed to substitute. The failure to substitute that emerged in the current research seemed peculiar to this research as it had not been reported in other researches mentioned in the reviewed literature. Although Makonye mentioned the problem of substitution, it was in the context of the functional notation
in which the student failed to understand $f(x + \delta x)$ as a mapping but rather multiplication as $f(x + \delta x) = f(x) + f(\delta x)$. Students in the current research had the correct understanding of the substitution but failed to execute it. This indicated that the students’ understanding of the concept of substitution is instrumental and therefore influenced students’ thinking in differentiation. This thinking however could be a result of the heap thinking as the student in her explanation could be heard saying "$f$ times $x$ is $x^2$", although it was not clear what she meant by this. Her thinking was also complex in nature as she used surface association to associate $a(a + b) = a^2 + ab$ to $(a + b)^2 = a^2 + b^2$. The student’s focus was on the signifier, multiplication (Sfard, 2000) rather bracket expansion. With association complex, the thinking relies on use of patterns, which in this case were wrongly perceived by the student. The student also failed to understand what the teacher was suggesting to her during the subsequent discussion suggesting a problem with the English language. The second student demonstrated an ability to think in concepts as she showed a good understanding of both substitution and differentiation from first principles as a limit. This form of association was also evident when students were asked to differentiate \[ \frac{x^2}{25} + \frac{y^2}{16} = 1, \] when they first took square roots and wrote \[ \sqrt{\frac{x^2}{25} + \frac{y^2}{16}} = \frac{x}{5} + \frac{y}{4}. \] This was done on one side, ending with a linear function which was differentiated to get a constant. Only 5% of the students handled this problem conceptually using implicit differentiation.

After giving the students tasks to differentiate $x^3$ and $x^4$, students managed to deduce the rule for differentiating power functions and used it to differentiate polynomials successfully. In some cases it was also noted that the students had challenges with
the use of notations as the majority failed to handle $\frac{d}{dx}(f(x))$ notation and used $\frac{dy}{dx}$ even where $y$ was not part of the problem which other researchers also reported (Makonye, 2011). In this study students had challenges with fractional powers. Some of the errors were careless errors such as $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{1}{2}}$. Makonye discovered difficulties with the use of the power function rule which he termed procedural extrapolation. This involved differentiating constants as well by applying the power function rule such as $\frac{d}{dx}\left(\sqrt{x}\right)^2 = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)^2 = \frac{1}{2}x^{-\frac{1}{2}} - 2^{-2}$. This type of thinking was not encountered in this research.

5.2 Differentiation of products and Quotients

For the question find $\frac{dy}{dx}$ if $y = (x^2 + 3x)(x^3 + 1)$, one student had the answer

$$\frac{dy}{dx} = (2x + 3)(3x^2)$$

$$= 6x^3 + 9x^2.$$

When probed further about her answer, she said that is what she ‘thinks’ since the function is a polynomial so it made sense to differentiate each term separately then take the product. She was asked to differentiate $y = (x^2)(x^3)$ by expanding and then using her ‘thinking’ and she realised that the answers were not equal. After discussing the result and the product rule introduced the class was given another task to find $\frac{dy}{dx}$ if $y = \frac{x+2}{x-1}$, which generated interesting responses from the pupils among them:

$$y = \frac{x+2}{x-1}, \quad \frac{dy}{dx} =$$

$x + 2 = u, \quad \frac{du}{dx} = 1$
\( x - 1 = v, \quad \frac{dv}{dx} = 1 \)

\[
\frac{dy}{dx} = u \frac{dv}{dx} - v \frac{du}{dx}
\]

\[
= x + 2(1) - (x - 1)(1)
\]

\[
= x + 2 - x + 1
\]

\[
= 1
\]

The following conversation ensued when the teacher asked the student to explain her working:

**Teacher:** Is the derivative a constant in this case?

**Student:** That is what I think.

**Teacher:** Ok. Let us go back to the original function. I have a few things that I want you to clarify. If we were to draw a graph of the function \( y = \frac{x+2}{x-1} \), is the graph a straight line in this case?

**Student:** I don't think so.

**Teacher:** Ok. Let us look at the rule you have used. What is the basis of your thinking?

**Student:** I am using the product rule sir.

**Teacher:** But is that the way we stated the product rule in the last lesson?

**Student:** No, it had a positive.

Teacher: So why are you putting a negative in this case?

**Student:** Because we are dividing so we subtract.
**Teacher:** So, it means when dividing we subtract...ok...what is the basis of this argument? Are you thinking in terms of laws of logarithms, indices or what......, what is it that made you believe that we have to subtract?

**Student:** That is what I think sir.

**Teacher:** I understand that part and I appreciate your thinking....but can you give the basis of your thinking...what made you think in that way.

**Student:** ......................... (No response)

**Teacher:** Ok. Thank you..... (Name)...Can someone comment on.........’s answer.  
**Yes...... (Name).....What do you think yourself?**

Two students pointed out that the answer was incorrect and therefore suggested the writing the function as\((x + 2)(x − 1)^{-1}\), then applied the product rule.

The student here is using heap thinking and complex thinking. It is heap thinking in that she regarded all division as referring to subtraction which is true for quotients or numbers in index for or in logarithm form. She did not pay attention to the context of the problem. It is also partly associative complex and partly collection complex. She is using collection complex because all functions involving division of products according to her, must be in the same group, they are similar, as the two reverse operations are treated as opposites by the student. It is associative complex centred on a nucleus of the product rule. Since that is what she had been introduced to, she used this past knowledge superficially (Berger, 2004). The student is basing her thinking on one trait in which the product and quotient differ and the relation to which they are functionally complementary (Vygotsky, 1986) cited in Berger (2004). The students showing this form of complex are therefore using contrasts to guide their thinking, since a collection
complex focuses on both similarities and contrasts. Tuomi (1998) referred to such thinking in which a student focuses on contrasts only as diffuse complex, which the student here displayed by looking at how a product and quotient differ. Berger however pointed out that collection complex does not usually translate into the mathematical arena. She nevertheless, gave an example of collection complex of differentiating taken as integration, which is similar in the case of the current research where division is taken to mean subtraction with the product rule. It is important to note that collection complexes can also be categorized as chain complexes, they (chain complexes) also rely on similarities and contrasts. On the other side, the students who handled the problem of differentiating a quotient well showed fully developed concept of indices, so were thinking in concepts, in particular scientific concepts.

**Question 4:**

Find the $x$–coordinates of the points on the curve $y = \frac{x^2 - 1}{x^2}$ at which the gradient of the curve is 5.

There were two dominant approaches to this task. Both methods showed that the students were either thinking in concepts or demonstrated relational understanding. Some students used instrumental understanding as they could not use the quotient rule without writing it down first. The other group who demonstrated relational understanding simplified the expression to: $1 - \frac{1}{x^2} = 1 - x^{-2}$, then used the power and polynomial functions rules to differentiate. When dealing with quotients, the type of thinking identified by other researchers in which $\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right)$ is given as equal to $\frac{f'(x)}{g'(x)}$ was not noted in the current study but rather a different form of thinking was discovered in which the student linked quotients to laws of logarithms and still applied the product
rule because division is the inverse process of multiplication. For the student the rule
\[ \frac{dv}{dx} + \frac{du}{dx} \] became \[ \frac{dv}{dx} - \frac{du}{dx} \] to differentiate \( \frac{x^2 + 2}{x^3} \). This is a form of associative complex thinking but incorrectly applied in view of an unfamiliar context. The form of thinking which other researchers have reported on involving derivatives of product of two functions was confirmed in this research as some students think that \( \frac{d}{dx} (f(x) \times g(x)) = f^1(x) \times g^1(x) \) as shown by the student who thought
\[
\frac{d}{dx} \left((x^2 + 3x)(x^3 + 1)\right) = (2x + 3)(3x^2)
\]
\[ = 6x^3 + 9x^2. \]

5.3 **Differentiating exponential and logarithmic functions**

The exponential function displayed other challenges and interesting forms of thinking from the students. The teacher started by asking students to differentiate \( a^x \). Two interesting solutions emerged:

1. \[
\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(f(x+\delta x) - f(x))}{\delta x}
\]
\[ = \frac{a^{x+\delta x} - a^x}{\delta x}
\]
\[ = \frac{\delta x}{\delta x}
\]
\[ = 1
\]

2. \[
\frac{d}{dx} (a^x) = x a^{x-1}
\]
Scaffolding revealed that in the first case the student did not understand the concept of substitution, use of brackets and the derivative as a limit, hence the omission of the limit as in the other previous cases. The thinking here is that once the substitution is done, the limit can be dropped. Although it made sense for the student to use differentiation from first principles, since students had no available method to deal with the exponential function, relational understanding influenced the thinking process, giving a constant gradient for such a function. Probing the student who gave the second answer resulted in the following conversation:

**Teacher:** Can you explain your answer?

**Student:** The derivative is correct because $a^x$ is a power function and I have used 

\[ \frac{d}{dx} (x^n) = nx^{n-1}. \]

**Teacher:** Fine. So let us consider $\frac{d}{dx} (x^n)$; when we established the result, what did we say about $n$?

**Student:** It must be a rational constant. It does not change.

**Teacher:** Good, a constant which maybe negative or positive. So is $x$ a constant in this case?

**Student:** No....i don't know...ah it is a variable sir.....ahhh ...sir ...yes my answer is not correct (student laughs).

Students were thinking in complexes. The first student is not yet thinking in concepts but seem to be using pseudo concepts as her concepts of substitution, and indices have not fully developed. Since she is relying on genuine features of concepts, then her thinking can also be categorized as diffuse complex. The second student could
not realise that the power in this case is a variable. Although scaffolding revealed that
the student has not fully developed the concepts of indices, rational numbers and
constant, she is also relying on pseudo concepts. In addition the student used surface
associative complex as she associated $x^n$ with $a^x$. To her the power forms the nucleus
and anything that has a power must be differentiated using the power function rule.
Since the power is a nucleus, the complex according to Berger (2004) is nucleus
centred associative complex. This is also a form of collection complex as the student
focuses on the power to group the function in the same class as $x^n, \text{for } n \in \mathbb{Q}$ and
therefore treat them in the same way despite the differences in the powers. Some
students however thought in concepts as they correctly used other concepts to arrive
at a correct answer:

$$\frac{d}{dx}(a^x) = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x}$$

$$= \lim_{\delta x \to 0} a^x a^{\delta x} - a^x$$

$$= a^x \lim_{\delta x \to 0} (a^{\delta x} - 1)$$.

The students thought using scientific concepts of factorisation, indices and
substitution. Although they failed to explain their result, it was evident that they used
the rule properly with an understanding that the derived function is a limit.

Students showed heap thinking when given tasks that involved the special exponential
function $e^x$ and $lnx$. Their thinking was focused on the presentation of the questions
in terms of sequencing. The derivatives:

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$
\[
\frac{d}{dx} (e^x) = e^x
\]

\[
\frac{d}{dx} (\ln x) = \frac{1}{x},
\]

influenced the thinking of the students when they were followed by exponential or logarithmic functions that were composite functions. For example, when asked to find \( \frac{d}{dx} (y^2) \) students gave \( \frac{dy^2}{dx} \) and \( \frac{d}{dx} (e^y) = e^y \). One student gave \( \frac{d}{dx} (\ln x) = e^x \) and \( \frac{d}{dx} (\ln(x + 2)) = \frac{1}{x+2} \) followed by

\[
\frac{d}{dx} (\ln(x^2 + 3x)) \text{ yielded } \frac{1}{x^2+3x}.
\]

This type of thinking is consistent with what Berger (2004) explained as thinking influenced by layout of a page, which is a form of heap thinking. The associative complex (surface) and collection complexes are inherent in this form of thinking. The students could not realise the need to use the rule for differentiating composite functions as well as implicit differentiation. The thinking shows that students believe that anything exponential or logarithmic (natural) must be differentiated the same as the special forms. This is an association complex as students rely on patterns. The student who gave \( \frac{d}{dx} (\ln x) = e^x \) used the relationship of a function and its inverse to imply that they have the same properties and therefore the derivatives are equal. Since she was introduced to the derivative of the exponential function first, she used heap thinking to link the two. This may have been due to failure to identify the right schema to use to deal with the new function, hence use of what she already knew that the inverse of \( \ln x \) is \( e^x \) and that \( \frac{d}{dx} (e^x) = e^x \). This is what Berger (2004) referred to by the example of a student who treats anything that lives in water as fish. Thinking was also confirmed in this research when students linked the properties of \( \ln x \) to the properties of \( e^x \). This was evident when the students were given the question.
(i) Given \( y = \ln x \), write four properties of the function

(ii) Find the derivative of the function and write the properties of this derived function.

The use of the properties of \( f(x) \) to derive properties of \( f^1(x) \) confirms the report by Berger (2004) as heap thinking.

**Question 5**

Differentiate the following with respect to \( x \)

(i) \( y = \ln(x^2+1) \)

(ii) \( \ln\left(\frac{x+2}{x+3}\right) \).

Nearly all students found part one manageable. They applied the rules for differentiating a natural logarithm and composite function to get the correct derivative. However the sequencing and form of the first part seemed to have led the majority to use heap thinking to deal with part (ii). Instead of using the notation \( \frac{d}{dx} \left(\ln\left(\frac{x+2}{x+3}\right)\right) \), they wrote \( \frac{dy}{dx} \) yet \( y \) was not part of the problem in this case. They found it demanding to deal with the result

\[
\frac{d}{dx}\left(\ln\left(\frac{x+2}{x+3}\right)\right) = \frac{1}{x+2} \times \frac{d}{dx}\left(\frac{x+2}{x+3}\right).
\]

Twenty five percent of the students found it convenient to take this as equal to \( \frac{x+2}{x+3} \times \frac{1}{x+3} \) suggesting that differentiating \( \frac{x+2}{x+3} \) just involves differentiating the numerator rather than the use of the product rule. Thus students displayed both heap thinking (use of sequence and form of problem) and associative complex (associated result of part (i) as a cue of what was to be done on part (ii). One student used collection complex as she thought all problems
involving natural logarithm must be equated to $y$, exponentiated then differentiated implicitly. The concepts she used are relevant to the situation but have not fully developed hence she used pseudo-concepts as shown by her solution: 

$$y = \ln \left( \frac{x+2}{x+3} \right)$$

$$e^y = \frac{x+2}{x+3}$$

$$e^y \frac{dy}{dx} = \frac{1}{(x+3)^2} \quad \text{(Used the quotient rule though not shown)}$$

$$\frac{dy}{dx} = \frac{1}{(x+3)^2} \frac{1}{e^y} = \frac{1}{x+3} \times \frac{x+3}{x+2}$$

$$= \frac{1}{x+2}.$$

Some pupils demonstrated relational understanding which determined a solution strategy which was not involving. This group of students were thinking in concepts as they used laws of logarithms and then differentiated the individual terms using the rule 

$$\frac{d}{dx} (\ln x) = \frac{1}{x}$$

ie. 

$$\frac{d}{dx} \left( \ln \left( \frac{x+2}{x+3} \right) \right) = \frac{d}{dx} (\ln(x + 2) - \ln(x + 3)).$$

The students demonstrated an understanding of the underlying concepts, and therefore this understanding influenced their thinking in differentiating the function.
5.4 Implicit differentiation

Question 6.

Find the equation of the normal to the curve $3x^2 + 4y^2 = 12$ at the point $\left(-1; \frac{11}{2}\right)$ expressing the equation in the form $ax + by + c = 0$, where $a, b$ and $c$ are integers.

Question 8.

The curve has equation $x^2 + xy + y^2 = 3$

(i) Show that $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$

(ii) Find the coordinates of the points on the curve where the gradient is zero.

A typical solution common with students was as follows:

[Image of a handwritten solution showing the steps of implicit differentiation for Question 6 and Question 8.]
On implicit differentiation, the major challenge was on notation. Most students’ thinking was associative complex and heap thinking. From the definition of differentiation, functions were stated explicitly as $y = f(x)$, and students thought each time they were asked to differentiate, they would use $\frac{dy}{dx}$, an indication of heap thinking. It appears the majority could not think of differentiation without using $\frac{dy}{dx}$. It is evident in this question when, instead of using the notation $\frac{d}{dx}$, the student just used $\frac{dy}{dx}$ and her statement lost meaning. The students’ thinking in this instance is still complex in nature as students used pseudo concepts. Incorrect notation and signs, despite good understanding of some concepts involved suggests instrumental understanding which greatly influenced the students’ thinking.

Artefact 2: Wrong answer due to incorrect signs

From the statement she wrote on the second part, the student knew the concept of product of gradients of perpendicular lines, but seemed not thorough on the execution of the idea as the working was marred by incorrect gradient sign. Differentiating $x - $
\( x \ln y = \ln y \) was really demanding for nearly all the students. Their thinking was characterized by diffuse complex and pseudo concepts. They realised the need for implicit differentiation and use of the product rule, but found it very challenging. The few that showed conceptual understanding, either failed to leave the answer in a neater form, or muddled their working with wrong signs.

5.5 Application of differentiation

Optimisation problems did not present much challenges as reported in other researches. The following were typical problems given to pupils on maximizing and minimizing values of functions using differentiation.

**Question 7**

Calculate the coordinates of the stationary points on the curve with equation \( y = x^2 - x \). Show that the stationary point is a maximum point.

**Question 13**

A rectangular open concrete tank has holding capacity 4000 \( m^3 \). The tank rests on a horizontal base and has a uniform square cross section. The walls and base are 1cm thick. The internal dimensions are \( xm \) by \( xm \) by \( ym \).

(i) Show that the volume \( V \) of the concrete used is given by

\[
V = 4y(x + 1) + (x + 2)^2
\]

(ii) Express \( V \) in terms of \( x \).

(iii) Hence find the value of \( x \) needed to minimize the amount of concrete used.

The majority of the students (71%) demonstrated that they understood optimisation as they displayed thinking in scientific concepts. For question 7, students successfully
differentiated and equated the derivative to zero to get $x = \frac{1}{4}$ at the stationary point.

The only challenge presented itself on the second part of the question on determining the nature of the turning point. Students managed to get

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}},$$
and concluded that since $\frac{d^2y}{dx^2}$ is negative, it shows ‘its’ a maximum. This thinking can be viewed as heap thinking as the student just used the sign of $\frac{d^2y}{dx^2}$ to make a conclusion, instead of evaluating $\frac{d^2y}{dx^2}$ at $x = \frac{1}{4}$. Question 13 presented language problems. The students’ answers indicated that they had developed concept of optimisation, but could not formulate the problem as they failed to interpret the question correctly. This also confirms the language barrier in learning of mathematics as pointed by Radatz (1979). Wrong solutions were also obtained as a result of failure by the students to manipulate terms in their simplifications, in particular the failure to factorise. This agrees with other researches’ findings when they attributed some forms of thinking in differentiation (as evidenced by the presence of errors) to other topics such as algebra and trigonometry.

When given word problems (which Makonye, 2011 classified as modelling problem) students showed language problems. As reported by Makonye, this research therefore revealed that optimisation problems were not accessible to pupils who did not realise that differentiation offered a special technique to maximize or minimize quantities. Such students were seen not able to think of differentiation beyond writing down symbols and notation used to communicate this differentiation.

The following solution (artefact 3) is another instant when a student show some form of pre concept thinking. For this student the thinking has not fully developed as she is confusing her substitution, using $u$ and $dv$ as in integration by parts. When asked why
she used such a substitution, she responded that because we need a $v$ in the formula, suggesting that she is thinking about integration. The fact that she wrote the product rule correctly showed that she had the knowledge, but the knowledge has not fully developed. The student was supposed to use the substitution $y = uv$, in which $u = x$ and $y = \sin(6x + \frac{\pi}{4})$ then use the product rule. This is consistent with what other studies have reported that many students have adequate procedural knowledge of differentiation but face problems when conceptual understanding is needed in problem solving situations. (Porter & Masingila, 2000; Orton, 1983b). In this case, the student has procedural knowledge on how to use the product rule, but her understanding of substitution and use of the product rule has not fully developed. Since the student could identify the important features of the problem, she is using diffuse complex, though she could not see how the notations for differentiating and integration differ.

The differentiation therefore appeared demanding for the student and she confused differentiation with integration. The student had the concept of the product rule but her substitution seemed to indicate ideas in integration. Since the student has the right concepts, but her use of the product rule and the subsequent substitution had not fully developed, she is also using pseudo concepts.
Artefact 3: Product rule with incorrect notations for substitutions.

5.6 Summary

This section therefore shows that students' thinking when learning differentiation is varied and unique. The thinking can however be classified into heap thinking, complex thinking and conceptual thinking. The heap thinking was usually influenced by sequencing of questions. Associative and collection complexes as well as pseudo
Chapter 6: Summary, conclusions and Recommendations

6.0 Introduction

This concluding chapter is the climax of the study to analyse students’ thinking when learning differentiation at advanced level. Guided by Vygotsky’s theory of concept formation, students’ written and verbal responses were analysed using content analysis. This chapter therefore concludes the research by summarising discussions, conclusions and making recommendations.

An analysis of students’ thinking when learning differentiation was premised on constructivism, that learners construct mathematical knowledge basing on what they already know. This is achieved by incorporating new mathematical ideas onto their earlier knowledge. Constructivism is the basis of Vygotsky’s theory of concept formation which explains how students think when they learn new mathematical concepts. According to the theory, students make use of heap thinking, complex thinking as well as thinking in concepts, which is basically relating what they already know to their new situation. Makonye (2011) noted that as students come to lessons with semi constructed realities of what we teach them, their constructed realities are important to the development of pedagogical content knowledge. It is however important to note that these constructed realities can be stumbling blocks for learning if teachers do not understand them as they can be potential misconceptions. Piaget (1968) cited in Makonye further asserts that students’ earlier knowledge can support or hinder students’ learning as they use it to make sense of new mathematical ideas through accommodation, assimilation and equilibration. Vygotsky concurs, but views...
the process, despite errors and misconceptions, as an inevitable process of concept formation, which every teacher must understand if students are to be taught mathematics effectively.

The findings, conclusions and recommendations of this research therefore arise in the background of the aforementioned theoretical assumptions while answering the following questions:

*What are the different ways of thinking displayed by students when learning differentiation?*

The sub-questions centred on the students' thinking were:

- *What form of heap thinking is used by ‘A’ level students when learning differentiation?*
- *What are the various complexes displayed by ‘A’ level students when learning differentiation?*
- *What are the other characterizations of students’ thinking when learning differentiation?*

### 6.1 Summary of the study

The study sought to determine and categorize students’ thinking when learning differentiation at advanced level. A case study design was used in which content analysis was done through reading pupils’ written work as well as listening to their verbal responses during task based interviews. In doing so, it was critical to understand what learners had said or written to determine how they were thinking in arriving at their answers. Such an analysis was aided by Vygotsky’s theory of concept formation, which, as a theoretical framework, provided categories for characterization
of learners’ thinking when learning differentiation. This characterization therefore resulted in grouping students’ thinking into heap thinking, complex thinking and thinking in concepts when learning differentiation. Although the three forms of thinking were noted when students learn differentiation, associative, collection complexes and pseudo concepts were seen to be the most prevalent sources of errors and misconceptions when learning differentiation. The literature review provided other characterizations outside Vygotsky’s characterization which focused more on the output (errors and misconceptions) rather than the process resulting into these. Such characterizations are however important to this study since heap and complex thinking are thinking forms resulting in errors and misconceptions.

6.2 Research findings

In this section, I report that the analysis revealed that each learner develops a distinct way of reasoning and thinking that may not be similar to the way other students think, yet that thinking is important in its own right. The results of the study therefore provided the following answers to the research questions posed in the study.

6.2.1 Forms of heap thinking when learning differentiation

Heap thinking was evident in pupils’ work on differentiation. It manifested itself when students related their algebraic knowledge to the function notation, \( f(x) \). Here, students believed that it is always true in the context of use of brackets for multiplication that \( f(x) = fx = f \times x \). This form of thinking confirms Makonye’s (2011) view that such heap thinking is due to failure to recognize the essence of symbolism of a function as a mapping of a point \( x \) in the domain to some point \( f(x) \) in the range. Heap thinking was also evident when students regarded differentiation of all quotients as division associated with subtraction and therefore used the product rule with a
negative sign. In their minds, such students associated all division with subtraction as in ‘subtraction of powers or logarithms’. This kind of thinking is what Berger (2004) referred to when she gave an example of heap thinking as treating everything that lives in water as fish. Although she indicated that this thinking does not translate itself to thinking in mathematics, it was evident in this research. Another form of heap thinking noted in this research is what Berger referred to as thinking that is influenced by page layout. In this instance students showed heap thinking when their answers were influenced by the presentation and sequencing of questions. When students were asked to differentiate \( e^{x^2-2} \) or \( \ln(x^2 + 3x) \) following the derivatives of \( e^x \) and \( \ln x \) they just gave the derivatives as

\[
\frac{d}{dx}(e^{x^2-2}) = e^{x^2-2} \quad \text{and} \quad \frac{d}{dx}(\ln(x^2 + 3x)) = \frac{1}{x^2 + 3x}.
\]

Heap thinking was also shown when students linked properties of \( \ln x \) to properties of \( e^x \) as well as deriving properties of \( f^{1}(x) \) from \( f(x) \), which confirms what Berger noted as a form of heap thinking.

### 6.2.2 Forms Complex thinking

Complex thinking appeared the most prevalent form of thinking when students learn differentiation. Since, in one way or the other pupils relate what they already know to something new, complex thinking manifested itself in the form of associative, collection and pseudo concept complexes. Surface association was evident in algebraic concepts in particular substitution and bracket expansion. For \( f(x + \delta x) = (x + \delta x)^2 \), the answer was given as \( x^2 + \delta x^2 \). Here students focused on the signifier(multiplication) as in \( x(y + z) = xy + xz \), rather than the process of bracket expansion \( (x + y)^2 = (x + y)(x + y) \). Collection complex was used when students grouped all functions that they supposed to be in the same class and treated them the
same when differentiating. The quotient \( \frac{(x+2)}{(x+1)} \) was taken to be in the same group as products and so the product rule could be used. Thinking this way made sense, but the students’ thinking then indicated that the product rule must be used with a negative since the quotient involved division. To the students multiplication is associated with plus (product) and division with subtraction (quotient) as opposite pairs of operations.

The product rule \( v \frac{du}{dx} + u \frac{dv}{dx} \) became \( v \frac{du}{dx} - u \frac{dv}{dx} \) to differentiate quotients. Use of the nucleus surface association was used to differentiate exponential functions with unknown in the power. By associating \( x^n \) to \( a^x \), \( x^n \) formed the nucleus, so that anything that has a power must be differentiated using the power function rule giving the answer \( x a^{x-1} \) to \( \frac{d}{dx} (a^x) \). This can also be viewed as a collection complex as all power functions are grouped together and differentiated using the rule.

Pseudo concepts also appeared common in this research. Concepts of substitution and indices seemed not fully developed for the students as they were constantly misused in students’ solutions resulting wrong answers. In some cases use of collection and associative complexes also involved use of pseudo concepts. When students associated all logarithm functions to the classification of functions in which the first step involves equating them to \( y \) then exponentiated both sides before differentiating implicitly, they showed the afore mentioned complexes. First students thought all logarithm functions must start with \( y \) (associative and collection complexes). Secondly the concepts of creating an equation, exponentiation and use of the notation \( \frac{d}{dx} \) had not fully developed in the students’ minds (pseudo concepts). It can therefore be summed up that the common complexes shown by students when learning differentiation are associative, collection and pseudo concept, with other complexes not seen to be used at all. In some instances students showed use of
diffuse complex when besides focusing on a single genuine feature of a problem, they used contrasts between problems to direct their thinking. This, for example, was shown when students focused on the difference between multiplication (product of two functions) and division (quotients), although they had identified important features of the problem to justify use of product rule.

6.2.3 **Thinking in concepts**

Apart from use of heap and complex thinking, students also demonstrated that they thought in concepts. Their understanding was relational and therefore such students used other concepts to successfully accomplish given tasks. Students who differentiated $\ln \left(\frac{x+2}{x+3}\right)$ by using

$$
\ln(x + 2) - \ln(x + 3) \text{ or } \frac{x^2-1}{x^2} \text{ using } 1 - \frac{1}{x^2} = 1 - x^{-2} \text{ or }
$$

$$
\frac{d}{dx} \left(\frac{x+2}{x-1}\right) = \frac{d}{dx} \left(\left(x + 2\right)\left(x - 1\right)^{-1}\right) \text{ demonstrated thinking using concepts, an indication of good understanding of underlying concepts.}
$$

6.2.4 **Other thinking forms**

The literature review also showed other forms of characterization of students’ thinking when learning maths in general and differentiation in particular. The classification however focused on errors and misconceptions (products) rather than thinking (process). However these classifications were important for this study as heap and complex thinking produces incorrect answers. Donaldson(1963) cited by Makonye (2011) classified these errors and misconceptions into

- Executive errors, which are a result of failure by students to carry out manipulations procedures even though required concepts would have been
understood (Orton, 1983a). These errors are a result of failing to carry out an algorithm. Students failed to differentiate due to algebraic executive errors such as failure to substitute, factorise of change of subject.

- **Structural errors**, which arose from failure to appreciate the relationship involved in the problem or group some principles essential to the solution. Makonye noted that these can be compared to conceptual errors which arise from mistaken perceptions about the nature of the mathematical concepts. These errors involve errors such as \( f(x + \delta x) = f(x) + f(\delta x) \) which arose from thinking that fails to appreciate the symbolism of function as a mapping rather than multiplication being distributed over addition.

- **Arbitrary errors**, in which students ignore part of available information while acting on the rest. Students behave arbitrarily and fail to take account of the conditions of the question, usually in an attempt to make problem to suit what they already know.

Hirst (2003) cited in Makonye (2011) proposed a rather different classification involving procedural extrapolation, pseudo linearity and equation was balancing. These arise out of learner’s predisposition to generalise what they already know to new situations.

When students gave answers such as \( \frac{d}{dx}(e^{2x}) = e^{2x} \) by using \( \frac{d}{dx}(e^{x}) = e^{x} \), they are using procedural extrapolation. This involves the students viewing the exponential function as an object rather than a process. Vgotsky called this associative complex thinking. Pseudo linearity involves erroneous extrapolation such as \((x + y)^2 = x^2 + y^2\), which resulted in answers such as

\[
    f(x + \delta x) = (x + \delta x)^2 = x^2 + \delta x^2 \quad \text{or} \quad \frac{d}{dx}(x^2 + 3x)(x^3 + 1)) = (2x + 3)(3x^2).
\]

Underlying such thinking is the distributive rule from algebraic simplifications such as
\( x(y + z) = xy + xz \). Such thinking is what Vygotsky termed complex thinking. According to Hirst, equation balancing emanates from the balance method of solving equations, when we do the “same thing to both sides of an equation”. In their thinking students replace “both sides of an equation” with “both sides”, resulting in thinking that produces answers such as \( \frac{d}{dx} \left( \frac{x+3}{x+1} \right) = \frac{d}{dx}(x+3) = 1 = 1 \). Although such thinking was not evident in the current research, other researchers have reported it in other researches on differentiation (Makonye, 2011).

Besides these classifications, thinking in mathematics has also been classified as intuitive thinking, analytical thinking, visual thinking, analogical thinking and integrated thinking. These forms of thinking were seen to be all used by pupils in learning differentiation, except visual thinking. Analytical thinking was prevalent as it involves symbolic and verbal representations as well as solution strategies that involve a sequence of steps. This involved situations when students could follow steps in differentiation, but because their understanding was procedural, they failed to think in concepts to show relational understanding. Pupils could also be seen using analogs to show knowledge transfer involving comparisons. They however used these transfers outside their domains, resulting in complex thinking and consequently resulting in errors. This is consistent with what Alwyn & Dindyal (2010) observed when they asserted that instrumental understanding give rise to analogical reasoning errors in which procedures are detached from their meanings in source systems and are in appropriately applied to other contexts. Without adequate understanding on both the source and target systems, students simply focus on the surface similarities and neglect the structural differences. It can therefore be concluded that whenever pupils used other forms of thinking such as intuitive, analytical, or analogical thinking, it
resulted in the three categories; heap thinking, complex thinking or thinking in complexes.

6.3 Conclusions

In their responses to mathematical tasks, students often show many different forms of thinking which demonstrates how they learn mathematical concepts. This research therefore revealed that students’ thinking can be heap thinking, complex thinking or thinking in concepts. The first two can manifest themselves as errors or misconceptions while thinking in concepts produces correct answers to given tasks. Nevertheless, heap thinking and complex thinking are thinking forms that are essential in the development of scientific concepts, which is a prerequisite for learning mathematics for understanding. Since learning involves making connections or establishing relationships either within already existing knowledge which to the learner appeared unrelated or between existing knowledge and new information, learning differentiation at advanced level therefore showed no exception as students used both heap and complex thinking as well as thinking in concepts. The most common complexes used were associative complex, collection complex and pseudo concept complex.

6.4 Recommendations

6.4.1 Implications for teaching differentiation

- Heap and complex thinking are inevitable in learning mathematics in general and differentiation in particular. These forms of thinking result in errors and misconceptions which teachers must view as opportunities for deepening their understanding of the students' difficulties, which is critical for further learning.
Errors in differentiation must therefore be viewed as a vehicle for learning which must be understood rather than replaced and eliminated.

- When teaching differentiation enough examples must be given so that pupils have enough examples to refer to so that they avoid associations which result in flawed thinking. When teaching the derivative of $e^x$, for example, the basic rule should be followed by many examples with varying powers to show students that it would be wrong to collectively treat all exponential functions or logarithm functions as having the same derivative.

- Care must be taken when the techniques of differentiation are being developed and questions asked (scaffolding) to help pupils avoid using heap thinking as sequencing of tasks and techniques influences students’ thinking. In this research, when students were given the task to find $\frac{d}{dx}(y^2)$ the sequencing influenced their thinking to give $\frac{d}{dx}(y^2) = \frac{dy^2}{dx}$.

- Before teaching the topic on differentiation, it may be necessary to revise algebraic concepts such as substitution expansion and factorisation, as these greatly affect learners’ thinking when learning differentiation through associative and collection complexes. In these instances correct rules are inappropriately used in new unfamiliar situations.

- Teachers should ask questions that encourage students to reflect and verbalise their thoughts. Such questions focus students’ attention on meta-cognition, helping them to become aware of their thinking processes that would otherwise go unnoticed. A case in point is when a student explained differentiation of a quotient using the product rule with a negative sign. The student reasoned that since we are dividing, we subtract. Questioning helped her realise that the gradient of such a function cannot be constant. By inquiring and listening to
students’ thinking strategies, teachers demonstrate that they value careful thinking and help students devise creative strategies for solving problems (Simon, 1986).

- The teaching of differentiation also revealed that teachers should allow students to guess and conjecture as well as allowing them to reason things on their own rather than showing them how to reach an answer. Allowing students to explore and discover in an open classroom atmosphere gives students the freedom to explore and experiment with their ideas. Such an appeal to students’ intuition can result in interesting and insightful solutions which can reveal students misconceptions, thereby enabling teacher to successfully help them.

- It is important to note that mathematics knowledge cannot be transmitted from the teacher to the student without their understanding; rather it must be dynamically reinterpreted, reorganized and reconstructed in each learner’s mind (Cobb & Bakersfield, 1995 and Hatano, 1996 cited in Makonye, 2011). Such reconstruction, reorganization and reinterpretation can manifest itself as heap or complex thinking or thinking in concepts as shown by students learning differentiation in this research.

6.4.2 Implications for further studies

This study was done using a case study research design. Although the design was appropriate to generate theory on how students think, testing this theory would require a different design which would allow the coverage of a wider geographical area. A longitudinal study to follow up on the students’ thinking as they apply differentiation in form six and final examination over a two year period would have been appropriate for this study. Other researchers can further explore this area under such a design as the current research was done under a limited time frame. Research of such a magnitude
can help designing teaching programmes that help learners minimise thinking that results in errors and misconceptions. Since this was not the aim of the current research, it is another area that can be explored on how learners’ thinking that results in errors can be minimized.
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Appendix A

The following questions were used in the current research to elicit responses from the students. Some questions were given as written work, while others were given as class work during lesson discussions.

1. Differentiate from first principles \( y = x^2 \)

2. Differentiate with respect to \( x \): \( y = (x^3 + 3x)(x^3 + 1) \)

3. Find \( \frac{dy}{dx} \) if \( y = \frac{(x+2)}{(x+1)}. \)

4. Find the following derivatives
   (i) \( \frac{d}{dx}(y^2) \)
   (ii) \( \frac{d}{dx}(a^x) \)
   (iii) \( \frac{d}{dx}(e^y) \)

5. Find \( \frac{dy}{dx} \) for the following functions
   (i) \( y = \ln(x + 2) \)
   (ii) \( y = \ln(x^2 + 3x) \)

6. Find the equation of the normal to the curve \( 3x^2 + 4y^2 = 12 \) at the point \((-1; 1 \frac{1}{2})\), expressing the equation in the form \( ax + by + c = 0 \), where \( a, b \) and \( c \) are integers.

7. Calculate the coordinates of the stationary point on the curve with equation \( y = \frac{1}{x^2 - x} \). Show that the stationary point is a maximum point.

8. A curve has equation \( x^2 + xy + y^2 = 3. \)
   (i) Show that \( \frac{dy}{dx} = -\frac{2x+y}{x+2y} \)
(ii) Find the coordinates of the point on the curve where the gradient is zero.

9. A curve is defined parametrically by the equations

\[ x = t^3 - 6t + 4, \ y = t - 3 + \frac{2}{t} \]

Find,

(i) The equations of the normals to the curve at the points where the curve meets the x-axis

(ii) The coordinates of their point of intersection

10. Find the coordinates of the point on the curve \( y = \frac{x^2 - 1}{x} \) at which the gradient of the curve is 5.

11. Differentiate with respect to \( x \)

(i) \( y = \ln(x^2 + 1) \)

(ii) \( \ln\left(\frac{x+2}{x+3}\right) \)

12. Differentiate the following with respect to \( x \)

(i) \( y = 2^x \)

(ii) \( \frac{e^{-x}}{x} \)

(iii) Find the coordinates of the turning point on the curve \( y = 2e^{3x} + 8e^{-x} \) and determine the nature of this turning point.

13. Given that \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \) is an equation of a curve, find

(i) \( \frac{dy}{dx} \)

(ii) the equation of the tangent to the curve at the point \( \left(3, \frac{16}{5}\right) \)

14. Given \( x - y\ln x = lny \), find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \).
Appendix B: Extracts of students’ work and presentations

\[
x^2 + \frac{y^2}{25} = 1
\]

\[
x + \frac{y}{4} = 1
\]

\[
x + \frac{y}{4} = 1
\]

\[
x + \frac{y}{4} = 1
\]

\[
4x + 5y = 20
\]

\[
4x + 5y - 20 = 0
\]

\[
dy = 4x + 5y - 20
\]

\[
dx = 4 + 5
\]

\[
= 9 \neq \frac{x + y}{4}
\]

\[
\frac{x - y}{\ln x} = \ln y
\]

\[
\frac{d}{dx} \left( \frac{x}{\ln x} \right) = \frac{d}{dx} \left( \ln y \right)
\]

\[
1 - \left( \frac{\ln x \cdot dy}{dx} + \frac{y}{x} \right)^2 = 1
\]

\[
1 - \frac{y}{x} = \ln \frac{dy}{dx}
\]

\[
\frac{dx}{dy} = 1 - \frac{y}{\ln x}
\]

\[
x - y = 1 \ln y \cdot \frac{dy}{dx} + \ln x \cdot \frac{dy}{dx}
\]

\[
\frac{dy}{dx} \left( 1 + y + \ln x \right)
\]

\[
\frac{dy}{dx}
\]