MISCONCEPTIONS LEARNERS ENCOUNTER IN SOLVING QUADRATIC EQUATIONS: A CASE OF 3 SECONDARY SCHOOLS IN ZIMBABWE

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APPROVAL FORM

The undersigned certify to have read and recommended to Bindura University of Science Education for acceptance, a research project entitled “Misconceptions learners encounter in solving quadratic equations: A case of 3 secondary schools in Zimbabwe” submitted by Mazhindu Lloyd in partial fulfilment of the requirements for the degree of Masters of Science Education in Mathematics.

Supervisor _____________________________________________

Programme Coordinator _________________________________

External Examiner _________________________________

Date _________________________________________________
DECLARATION

I declare that: “Misconceptions learners encounter in solving quadratic equations: A case of 3 secondary schools in Zimbabwe”, is my own work and has not been submitted before for any degree or examination in any other university and that all the sources used or quoted have been indicated and acknowledged as complete references.

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Date: October 2016

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CHAPTER 1
BACKGROUND TO THE STUDY

1.1 Introduction

Quadratic equations and functions serve as the transition from linear functions to higher order polynomials and solving them may present challenges to students. It is important to study the errors, misconceptions and challenges learners have so as to improve their performance in mathematics. This chapter is concerned with the background and purpose of the study, the problem statement is discussed, the significance of the study is explained and the research question posed.

1.2 Background of the study

Low performance in mathematics at Ordinary Level in Zimbabwe is of growing concern for policy makers as well as educators (Mupa, 2015). In Zimbabwe the pass rate for 2015 was 26.17% for the calculator version and for 2014 it was 21% with mathematics recording the lowest pass rate in the two years for science subjects (ZIMSEC, 2016).

Table 1.1 Pass rate for science subjects (Zimsec, 2016)

<table>
<thead>
<tr>
<th>SUBJECT</th>
<th>YEAR</th>
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<tbody>
<tr>
<td></td>
<td>2014</td>
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<td></td>
<td>Pass rate %</td>
</tr>
<tr>
<td>Mathematics 4028</td>
<td>21</td>
</tr>
<tr>
<td>Geography 2248</td>
<td>41.11</td>
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<tr>
<td>Integrated Science 5006</td>
<td>21.90</td>
</tr>
<tr>
<td>Biology 5008</td>
<td>57.16</td>
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<tr>
<td>Physical Science 5008</td>
<td>61.23</td>
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<tr>
<td>Physics 5058</td>
<td>89.18</td>
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<tr>
<td>Chemistry 5071</td>
<td>55.95</td>
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<tr>
<td>Human and Social Biology 5097</td>
<td>38.38</td>
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In 2011 the percentage pass rate for mathematics (4028) was 13%, 17% in 2012 and 23% in 2013. All other subjects had a pass rate above average (Matabvu, 2015). Persistent poor performance of learners in mathematics necessitates a study of learners’ misconceptions in specific topics and quadratic equations have been chosen for this study.

Poor performance in mathematics is also a global phenomenon. As pointed out by Jackson and Wilson (2011), research has also shown that even in the United States of America teachers are complaining about poor performance in mathematics (as cited in Mupa, 2015). Quadratic functions and equations is one area where students’ misconceptions can be addressed and this helps to boost their performance. A strong base helps students assimilate knowledge easily and get better chances of performing well at higher levels.

In Africa mathematics development is a story of unfulfilled potential as cited in Mathematics in Africa: Challenges and Opportunities. A report to the John Templeton Foundation from the Developing Countries Strategy Group International Mathematical Union (2009). Ways need to be sought to assist learners as there is are a lot of challenges in mathematics education in Sub-Saharan Africa with poorly resourced schools, large classes and a curriculum hardly relevant to the daily lives of students. Education for all policy has resulted in a growing student population World Bank Working Paper No. 101 (2007) and by embarking on this research ways will be sought to assist learners with such limitations in solving quadratic equations.

As a mathematics instructor of over 10 years I have noticed the recurring errors that students make when solving quadratic equations by factorization, the quadratic formula or the graphical approach. Research into students’ conceptual difficulties is expected to be useful because conceptual difficulties result in errors (Radatz, 1980). Errors are not simply the result of ignorance or carelessness but are a result of weaknesses in cognition (Radatz).

Hansen (2006) defined errors as mistakes learners make when solving problems that may be caused by carelessness, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical topic, learning objective, concept, lack of awareness or inability to check the answer given or the results of misconceptions. Misconception is the misapplication of a rule, an over- or under- generalization or alternative conception of the situation. Errors can be non-systematic in which they have no apparent pattern to be identified
and are random and due to some other reasons such as forgetfulness, stress or carelessness. They
are not connected to other concepts (Egodawatte, 2011).

Nesher (1987) identified errors as being systematic when it is a line of thinking that causes a
series of errors all resulting from an incorrect underlying pattern, rather than sporadic
unconnected errors. Assumption is that when the same error or similar errors occur more than
once in different situations then it is possible that the student has a misconception and therefore
they are worth analyzing (as cited in Egodawatte, 2011).

By undertaking this study instructional programs can be designed to address students’ systematic
errors. Towbridge and McDermott (1980) stated that:

We feel that there is need for similar research on student understanding of other
important concepts in physics as in other sciences as well. The number of major
conceptual difficulties identified in this study proved to be relatively small and not
unique to an individual. We believe a similar situation prevails for topics other than
kinematics. Many students struggle over the same hurdles in the same sequence in
learning the same material. Thus descriptive analysis of conceptual understanding is not
only feasible, but is likely to be widely applicable. (p.1028).

For many instances I have noticed that in solving for example the equation \(5x^2 - 6x - 3 = 0\) some
students would put \(-6 \pm \frac{\sqrt{-6^2-4(5)(-3)}}{2(5)}\) leading to \(-6 \pm \frac{\sqrt{96}}{2(5)}\) and thus there are some
errors that would have been made in such a solution and errors of short division were also
observed on many candidates work (Zimsec, 2015). In being explicitly instructed to use a graph
to solve a quadratic equation after they have drawn the graph I have noticed that some students
would want to use an algebraic solution. Students tend to rely heavily on algebraic solutions and
would want to revert to algebraic solution if asked to use a graph to solve an equation. The
purpose of this study is why such misconceptions occur and to unravel other misconceptions
associated with solving quadratic equations.

Students’ misconceptions about quadratic functions have also concerned some researchers like
Parent (2015) who observed that students tend to think about isolated parts of the problem when
solving quadratic problems and relied much on procedural strategies. Results have also shown
that students preferred to work with the standard form rather than the vertex form when solving problems on quadratics and also preferred to algebraically solve a problem versus tabular or graphical strategies. According to Parent students may not have a profound understanding of graphs.

Students’ errors and misconceptions and choice of strategies in solving quadratic equations will be examined and teaching strategies that will help them deal with the errors and the misconceptions can be developed by engaging on a study of students’ misconceptions (Parent, 2015).

By researching on the misconceptions that students have about solving quadratics equations, there will be better understanding of their mathematical thinking and abilities and thus teachers will be better equipped to address these misconceptions in teaching. Justification of which points to emphasize in teaching will be sought and this helps in future curriculum planning as suggested in similar studies by Parent (2015), Didis and Erbas (2015), Vaivuytjamai, Ellerton and Clements (2006) and Ibeawuchi and Ngoepe (2012).

One of the studies on student errors and misconceptions in solving quadratic equations in Africa was by Makonye and Nhlahla (2014) and they observed errors and misconceptions because of inappropriate schema to solve problems as students held on to simple equations schema. Their schema should be assimilated to solve quadratic equations.

According to Olivier (1989) it is important to analyse students’ errors as they form part of a learners’ conceptual structure and teachers should find ways to teach a mathematical concept based on the learners’ misconceptions. According to Smith, Sessa and Roschelle (1993) learners may not be reluctant to give up their misconceptions as they have a meaning to them and they have constructed them (as cited in Makonye & Nhanhla, 2014).

Students’ errors have been attributed to carelessness, no understanding at all, confusing different concepts or failing to transition from process oriented to object-oriented thinking (Schoenfeld, 1986). Schoenfeld also pointed out that students’ superficial understanding of important concepts prevent them from applying proper algorithms or strategies. Resnick et al. (1989) attributes learning difficulties from failure to understand the concepts on which procedures are based.
1.3 Purpose of the study

Aim of the research is to identify the errors, misconceptions and challenges students have in solving quadratic equations. As posited by Crouch and Mazui (2001) and Hake (1998) it is a step to improve instruction in this area because if instructors have a good understanding of the conceptual difficulties students commonly have then they can potentially design learning activities which can effectively help students surmount those difficulties (as cited in Rowland, n.d. p. 7).

It is important to explore and research on students’ misconceptions to better understand students’ thinking and patterns in which they learn. The goal of the study is to examine why so many students fail to learn mathematics and this will be done by examining their errors and misconceptions in solving quadratic equations.

The purpose of the study is to discover the reasons for the errors. As posited by Pickthorne (1983) cited in Makgakga (n.d) most teachers appear not to fully realise the nature of their learners’ confusion. They are unable to discover the reasons why the errors are made by the learners. Most of the teachers identify learners’ errors but rarely diagnose them (Luneta, 2008). This study will attempt to identify the root cause of the errors and how best they can be corrected to benefit both learners and teachers.

1.4 Significance of the study

According to the National Research Council (NRC, 1989) mathematics is important for both individuals and the nation. For individuals it opens doors to careers and enable them to make informed decisions as mathematics is important in everyday life and has many applications (as cited in Li, 2006). For the nation it provides knowledge for technological advancement and the economy and Zimbabwe is also encouraging advanced level students to take Science, Technology, Engineering and Mathematics (STEM) subjects (“Science, Technology, Engineering and Mathematics [STEM] craze hits Zim”, 2016). According to the article, “STEM craze hits Zim”, sustainable socio-economic transformation is envisaged to be driven by investing in STEM disciplines and industrialization of the economy and employment creation will result if students take STEM subjects.
According to Poincare (1952) there is research which points out that students keep making errors even with “good” teaching in terms of reasonable standards (as cited by Sfard, 1991). Misconceptions about a concept affect further learning due to the hierarchical nature of mathematics knowledge, therefore it is necessary to change students’ early misconceptions for example ones in solving quadratic equations. Ibeawuchi and Ngoepe (2012) emphasized that conscious efforts must be made to identify learners’ misconceptions so that appropriate corrective measures can be taken.

Rich subject matter together with the knowledge of learners’ conceptions, preconceptions and misconceptions is the knowledge that teachers need to try and prevent the continued underachievement in performance of learners in mathematics. Teachers’ knowledge of learners’ misconceptions will help the teacher to plan effective instruction and assist learners to develop conceptual understanding. Teachers will be better equipped to assess learning. Researchers such as Hill, Schilling and Ball (2004), Hill et al (2009), Ball, Bass and Hill (2004), Adler and Davis (2006), Shulman (1986) and Marks (1990) have identified teachers knowledge of their learners’ thinking (conceptions and misconceptions) about the concept as an integral component of teachers mathematical knowledge for teaching and other components include subject matter knowledge, representation and instructional strategies (as cited in Ibeawuchi & Ngoepe, 2012).

According to Viri (2003) who concur with Hope and Townsend (1983) that if learning is viewed from a constructivist perspective then teacher’s knowledge of the learners’ cognition is the most important part of the teacher’s knowledge (cited in Ibeawuchi & Ngoepe, 2012). The teacher has to consider the learners thinking of the subject and the possible difficulties the learners will come across the topic and there is need to restructure the content for meaningful communication with the learners. Fennema, Carpenter and Carley (1983) further suggested a research based classroom instructional model (cited in Ibeawuchi & Ngoepe).

The study will contribute to the mathematics teachers’ knowledge base on how students develop conceptual understanding of the solution of quadratic equations. Classroom instruction will be influenced in the area of quadratic equations through knowledge generated about students understanding of solving quadratic equations. Schools and classrooms can be furnished with informed best practices.
By learning to solve some aspects of quadratic functions and equations this gives students the capability to explore and solve issues in the real world. According to Afamasaga-Fuatai (1992), Curran (1995) aspects of quadratics are used later in higher mathematics classes especially when dealing with higher polynomial functions as well as students’ lives once they leave secondary school (as cited in Parent, 2015). According to Saglam and Alacaci (2012) quadratic equations are considered important in school mathematics as they serve as a bridge between mathematical topics such as linear equations, functions and polynomials (cited in Didis & Erbas, 2015).

Quadratic functions are involved in describing the path of projectiles so being able to solve quadratic equations will assist learners to solve problems on projectiles (Brown et al. 2007). They also appear and are important in the design of suspension bridges, cross-section of automobile headlights, satellite dishes and radio telescopes and being used by the military in predicting where artillery shells will hit the earth and if pupils understand them they will be able to model the world around them ( Brown et al, 2007). According to Budd and Sangwin (2004) quadratics are also used to describe the orbits along which planets move and are a link with acceleration as they appear in equations of motion (as cited in Parent, 2015). The U shape of a parabola describe the trajectories of water jets in a fountain and hence useful in their design (“Applications of Quadratic Functions ”, n.d, para. 1).

The popular android game application Angry Birds use the concept of a parabola and to master it one has to understand properties of a parabola. According to Ridley (n.d) it is an iPhone game later ported to other platforms (cited in “Angry Birds and the Parabolic Instinct in Humans”, 2011). The physicist Dr Rhett Allain studied the game and pointed out in his article, The Physics of Angry Birds that the birds travel without air resistance and their path is a parabola (as cited in Chartier, 2012). There are as well a host of other games with the same phenomena of paths of objects that trace a parabola such as Pocket Tanks and Kitten Cannons (“Angry Birds and the Parabolic Instinct in Humans”). In the design of the game, equations of a parabola are really useful (“Parabolic movement to an image”, 2012).

In business quadratics can be used to model profit and hence determine maximum profit as we commonly use quadratic functions in situations where two things are multiplied together and they both depend on the same variable- quantity of a product sold. Revenue is represented as a product of price and quantity sold (“Applications of Quadratic Functions”, n.d, para. 1).
Teachers should be clear about errors and misconceptions students make when solving quadratic equations and how often students tend to make them, where the errors are from and how the errors could be remediated. Not being cognizant of students’ misconceptions in these concepts could hinder teachers in using proper strategies to help students (Li, 2006). As Brown and Burton (1978) pointed out” one of the greatest talents of teachers is their ability to synthesize an accurate ‘picture’ or model, of a student’s misconceptions from the meagre evidence inherent in his errors” (as cited in Li, p.12). As a result detailed information about students’ misconceptions in learning contribute to teachers’ classroom practice.

This study extends research and deepens understanding of students’ conception of solving quadratic equations and contributes to teachers’ pedagogical content knowledge and informs curriculum planning. According to Radatz (2011) such a research on misconceptions clarifies some questions on mathematics learning and examines students’ conception.

1.5 Statement of the problem

This study extends research on students’ error patterns in solving quadratic equations and what misconceptions underlie those errors even after instruction. Riccomini (2005) asserted that most teachers are unaware of mathematical misconceptions held by their learners (as cited in Luneta & Makonye, 2010). Nester (1987) further points out that teachers tend to “teach mathematics in line with its logical structure, quite oblivious of the need to balance the psychological standpoint from which learners ascribe their mathematical meanings” (cited in Luneta & Makonye).

As students have difficulties learning mathematics research on students’ systematic errors on a topic or concept can provide a good lense to examine the causes. Examining students’ wrong answers provides one way to demonstrate students’ understanding of a concept. On the other hand students’ correct answers may not necessarily indicate a good conceptual understanding of related knowledge as students may have memorized the procedure or definition without true understanding. Students’ correct answers are generally uniform which does not provide an appropriate research setting. “Research on students’ errors make it possible to identify specific deficits in the way students’ knowledge is connected so that instruction can be designed to address specific connections students lack or to point out why certain connections are
appropriate” (Hiebert & Carpenter, 1992, p.89). This study will attempt to explain the source of the wrong answers.

Even basic mathematics concepts or operations like whole number addition and subtraction may involve extremely complicated cognitive processes (Schoenfeld, 1985). He also pointed out that teachers are already familiar with these and tend to ignore or underestimate their complexity and thus take a naïve approach to teaching mathematics concepts or operations and this study highlights areas or connections to be paid attention to when students learn to solve quadratic equations.

1.6 Research question

This study will investigate the following question: The nature of students’ errors and misconceptions in solving quadratic equations and their origin.

1.7 Definition of terms

**Concept**–as defined by Sfard(1991) is a theoretical construct of a mathematics idea. It is an association of ideas interlinked to create a mathematical object.

**Conception** – is the whole cluster of internal representations and associations evoked by the concept when learners encounter it (Sfard, 1991).

**Errors** – as defined by Hanen (2006) errors are mistakes learners make when solving problems that may be caused by carelessness, misinterpretation of symbols or text, lack of relevant experience or knowledge related to that mathematical topic, learning objective, concept, lack of awareness or inability to check the answer given or the result of misconceptions. Errors can be non-systematic in which they have no apparent pattern to be identified and are random and due to some other reasons such as forgetfulness, stress or carelessness. They are not connected to other concepts (Egodawatte, 2011).

Nesher (1987) identified errors as being systematic when it is a line of thinking that causes a series of errors all resulting from an incorrect underlying pattern, rather than sporadic unconnected errors. Assumption is that when the same error or similar errors occur more than
once in different situations then it is possible that the student may have a misconception and therefore they are worth analyzing (as cited in Egodawatte, 2011).

**Misconceptions** – are strongly held stable cognitive structures which differ from expert concepts and affect in a fundamental way how students understand mathematical concepts and procedures and must be overcome, avoided or eliminated for students to achieve expert understanding (Hammer, 1996). Misconception is the misapplication of a rule, an over- or under- generalization or alternative conception of the situation.

According to Green, Pel and Flowers (2008), Nesher (1987) and Riccommini (2005) errors and misconceptions are related but different. An **error** is a mistake, slip, blunder or inaccuracy and a deviation from accuracy. According to Riccommini (2005) unsystematic errors are unintended, non-reccuring wrong answers which learners can readily correct by themselves. Systematic errors are recurrent wrong responses methodically constructed over space and time and are symptomatic of a faulty line of thinking causing them to be referred to as a misconception (as cited in Luneta & Makonye, 2013).

1.8 Delimitation

The study was conducted at 3 secondary schools in Zvimba District, Mashonaland West province because of costs and time and results may not be extended to other schools or districts in Zimbabwe.

1.9 Limitations

Sample chosen is a convenience sample and not a random sample then the results of the study cannot be generally applied to a larger population. A case study was chosen which looked at the problem in depth and hence results can be applicable to similar settings. Another limitation is that affective factors which include anxiety, motivation and confidence may have an effect on performance. Anxiety was reduced by maintaining anonymity and not writing the names of participants on questionnaires and explaining that the results will not be used to grade them.

1.10 Assumptions

For the research to produce consistent results it was assumed that the quality and time of instruction the students received was the same. It was assumed the participants made a sincere effort to complete the test items. It was assumed the participants answered the questions to the
best of their ability. Anonymity and confidentiality was preserved and participants were informed that they can withdraw with no ramifications on their part to reduce anxiety.

An ontological assumption underlying the study was that many students’ errors in mathematics are not simply the result of ignorance or carelessness, but are in fact systematic and common to a significant number of students across a wide range of contexts. Since the errors are systematic, research on a group of students can be used to identify what errors and conceptual difficulties are common.
CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction
This chapter will start by a review of theories that informs this study, then some related studies on misconceptions in solving quadratic equations.

2.2 Theoretical framework

2.2.1 Constructivism
Constructivism and its corollary theories of the concept image and concept definition form the theoretical framework. The constructivist theories as cited in Battacharya and Han (2001) have a history of starting in the 18th century with the philosophies of Immanuel Kant and Giambattista Vico. Cognitive constructivism a division of the constructivist theories rooted in its origin in the work of Piaget (1968) will be used to explain learners’ misconceptions. The cognitive constructivist theory of learning (Piaget, 1968; Von Glasserfeld, 1990; Hatano, 1996; Smith, Disessa & Roschelle, 1993; Siegler, 1995) is useful to explain and predict how learners conceive mathematical ideas and their misconceptions (Luneta & Makonye, 2013).

Constructivism holds that learning involves learners’ reconstruction of knowledge in their minds by giving meaning to ideas existing outside themselves and internalising them (Luneta & Makonye). Hatano (1996) viewed constructivism as each person constructing knowledge by himself/her self through a cognising mind driven by self-regulation. Cognitive conflict a state of perturbation results when individuals encounter circumstances at variance with their current understanding and this drives individuals to explore the problem in relation to their prior understanding. By thinking systematically and carefully to deal with the cognitive conflict, reconciliation will result in learning.

Von Glasserfeld (1989) asserted that in constructivism knowledge is not transferred from teachers to learners, it is actively constructed by a thinking student and students adapt and reorganise the experienced world with his/her mind (as cited in Luneta & Makonye, 2013). Furthermore Von Glasserfeld points out that with the constructivism theory learning is viewed as the capability of an individual to change his/her conceptual structure in response to perturbation
The teacher assists the learner by providing physical or mental models on which the learner can impose and abstract mathematical meaning. At its tenet is the argument that knowledge cannot be transferred undigested from teacher to learners but must be restructured and re-organised by each individual learner to construct meaning.

The learner is viewed as an active participant in the construction of his own knowledge. This construction activity involves the interaction of a child’s existing ideas and new ideas are interpreted and understood in the light of that learner’s own current knowledge as cited in Olivier (1989). Knowledge is organised and structured into large units of interrelated concepts. The unit of interrelated ideas in the learner’s mind is called a schema. A schema is the mental representation of an associated set of ideas and/or actions. A schema can be discrete and specific, or sequential and elaborate. It is invoked on encounter with the concept. Learning involves interaction between the existing schema and new ideas.

According to Siegler (1995) social factors though important are peripheral to the learning process in cognitive constructivism. Key mechanisms for learning are assimilation, accommodation, and equilibration (as cited in Luneta & Makonye, 2013).

In assimilation as cited in Olivier (1989) a new but familiar idea is encountered and incorporated directly into an existing schema that is very much like the new idea. The new idea is interpreted or recognised in terms of an existing schema. Existing concepts are expanded by the new idea and new distinctions are formed through differentiation.

In accommodation as cited in Olivier (1989) the existing schema needs to be re-constructed and re-organised as it is relevant but inadequate to assimilate the new idea which is quite different. Accommodation involves changing pre-existing schemata to adapt to a new situation. Internal mental structures are changed to provide consistency with external reality. Existing schemas or operations are modified or new schemas are created to account for the new experience. Such reconstruction leaves previous knowledge intact, as part or subset or special case of the new modified schema. The previous knowledge is never erased.

Assimilation is an active incorporation of an experience into a representation already available to the learner. Misconceptions often occur in assimilation through over-generalisation. When a learner meets a new mathematical object, he/she might think that the mathematical object belongs to
a class which he/she already has and operates on it as he does an object in that class (Luneta & Makonye, 2013).

Cobb and Bauersfeld (1995) argued that a learners’ construction of knowledge is through the use of prior mathematical knowledge and social interaction that generates contradictions (as cited Luneta & Makonye, 2013). Social interaction such as discussion with other individuals or reading something that does not fit into one’s conceptual structure give rise to a state of disequilibrium which leads learners to question their beliefs and to try out new ideas. As learners try to resolve these conflicts they become aware of their activity and construct increasingly sophisticated systems of thought. In this way cognitive structures accommodate experience which results in learning. Piaget (1968), Hatano (1996), Sfard (1992) pointed out that learners acquire mathematical knowledge through construction of more powerful mental structures, concepts, or logical structures (cited in Luneta & Makonye). A cognitive conflict, a state of mental unbalance or disequilibrium occurs when new information does not fit or is contradictory with the existing knowledge is encountered. To return to a state of equilibrium, the construction of new cognitive structures is needed and this process is called equilibration. This state must be present for cognitive development and learning to take place. Equilibration involves both assimilation and accommodation, Bhattacharya and Han (2001).

Constructivism is an appropriate perspective in the study, as errors and misconceptions are the natural result of individual learners’ cognitive efforts of their own sense-making in the face of new mathematical challenges. It explains how and why errors occur. It is important to understand the nature of misconceptions in order to minimise or repair the barriers and misconceptions represented in the learning of mathematical concepts and in this way lasting solutions can be found to the mathematics learning problems (Luneta & Makonye, 2013).

Smith et al (1993) and Nesher (1987) argued that there is a strong link between constructivism and learners’ mathematical misconceptions (cited in Luneta & Makonye, 2013). Ernest (1991) wrote, “Constructivism accounts for the individual idiosyncratic construction of meaning, for systematic errors, misconceptions, and alternative conceptions in the learning of mathematics” (p. 2). Confrey (1990) also argued that teachers do not teach learners the misconceptions they make, rather learners make the misconceptions by themselves. Constructivism explains errors and misconceptions learners make when they think about concepts. Confrey summed it up as,
“Misconceptions are taken as the strongest piece of evidence for the constructive nature of knowledge acquisition, because it is highly unlikely that learners have acquired them by being taught” (p.201).

Constructivism accounts for partially remembered and distorted rules as they are caused by rote teaching (Olivier, 1989). Ideas are memorised in rote learning as the learner fails to link a new idea to an existing schema. The idea is so different from an available schema such that assimilation and accommodation are impossible. Such isolated knowledge is difficult to remember. In order to understand an idea it must be incorporated into an appropriate existing schema.

Constructivism help to explain that learners are consistent in their formulation of misconceptions as there is a pattern in their thinking and the pattern may not be immediately apparent to the teacher. Analysis of learners’ work helps to reveal that pattern (Luneta & Makonye, 2013).

According to Olivier (1989) a learner can fail to solve a problem because he may not possess the schema that is needed or he may possess an appropriate schema but the retrieval mechanism cannot locate it. Other causes are that the retrieved schema is flawed or incomplete or an inappropriate schema is retrieved.

Constructivism also helps to explain malrules as learners often degenerate into distorting rules to overcome an obstacle.

2.2.2 Concept image and concept definition

Tall and Vinner’s (1981) idea of concept image and concept definition will shed more insight on learner’s errors and misconceptions. Concept image is referred to as a cognitive chunk of ideas that include all the mental pictures, set of properties that a learner has formed in his/her mind regarding all aspects of a specific concept (Luneta & Makonye, 2013). It is a framework a learner has created and developed as a result of personal experiences with a particular concept. According to Luneta and Makonye concept images can be correct partially correct or erroneous and are dynamic and constantly changing as learners think about and refine their concepts. Learners build and modify their existing concept images.

Unconscious concept images may be responsible for the errors that learners commit and when concept images differ with formal mathematical knowledge the consequences are
misconceptions and weak ideas that do not help learners successfully solve mathematical tasks (Luneta & Makonye, 2013).

Tall and Vinner (1981) defined a concept definition as the words specifying the concept as defined by the mathematics experts and it can be learnt instrumentally or relationally. Learners need to be presented with varying representations or models of the same concept to form a complete picture of the concept in their minds. Partial concept definitions cause learners to form concept images that do not match the concept definition (Luneta & Makonye, 2013).

2.3 Understanding mathematics: Procedural and conceptual knowledge

Herbert and Lefevre (1986) cited in Makgakga (n. d) defined conceptual knowledge as “knowledge that is rich in relationship, that can be thought of as a connected web of knowledge, a network in which linking relations are as prominent as the discrete pieces of information”, (p.3-4). It plays a more important role in the learning of mathematics than procedural knowledge. It is essential for learners to have conceptual understanding as in its absence they will ineffectively indulge in problem solving and follow wrong or unnecessary procedures to solve the problems. The way learners think about a concept depends on the cognitive structures learners have developed previously (Battista, 2001). Battista also indicates that if learners do not develop concepts by themselves they will have a narrow understanding of those specific concepts and they will not be able to engage in problem solving.

Procedural knowledge is the ability of learners to use a relevant procedure in solving mathematical problems by following the rules, methods and procedures in different representations. Learners may grasp relevant procedures but fail to use them correctly in solving mathematical problems (Siegler, 2003). If a learner has both procedural and conceptual knowledge he/she can solve more complex problems of the same concept (Siegler).

According to Nesher (1986) learners with conceptual knowledge produce substantial gain in both kinds of knowledge but those with procedural understanding produce substantial gain in procedural knowledge but less in conceptual knowledge which will impede learners’ growth in mathematics. Nesher also supports the view that if learners are only shown procedures to solve a particular problem without understanding the concept it is unlikely that such a learner would be in a position to solve more complex problems independently. If problems are difficult to solve or
seem unfamiliar then learned procedures may not help and will need a learner to have conceptual knowledge (Nesher).

Confrey (1990) posits that conceptual knowledge provides and constrains those procedures to be followed in solving mathematical problems. Siegler (2003) also asserted that if a learner does not have a good understanding of a concept it will result in using procedures of solving problems inappropriately. If a learner does not understand the concept it would be difficult for him/her to articulate the procedures to solve problems based on the concept (Battista, 2001)

2.4 Quadratics

2.4.1 Quadratic functions

This section will discuss the mental structures and strategies that a student might need in order to solve quadratic equations by a graph, factorization, the quadratic formula or by completing the square.

Quadratic expressions are expressions of the form $ax^2 + bx + c$ where $a$, $b$ and $c$ are constants and $a \neq 0$ . Quadratic functions are most commonly defined in standard form as $f(x) = ax^2 + bx + c$ where $a \neq 0$ . There are different ways to express the quadratic function. It can be also be expressed in factored form as $f(x) = a(x - x_1)(x - x_2)$ and the vertex form $f(x) = a(x - h)^2 + k$ . The quadratic function is called a quadratic equation when each of the forms is set equal to zero which gives the standard format $ax^2 + bx + c = 0$ . The “solution” of the equation is when “$x$” is solved. This is done through completing the square, the quadratic formula, factorizing and graphing. By graphing solutions are called roots, zeros or $x$- intercepts and are essentially when the graph intersects with the $x$-axis. The vertex of the parabola is the turning point of the graph and is a minimum if $a > 0$ and is a maximum if $a < 0$.

Crucial components in understanding the quadratic function is for students to learn how to read the graph. By learning how to read the graph and the different components of the graph they will heighten their understanding of functions in general. This will be helpful when they move on to calculus. By understanding the quadratic functions and finding the minimum, maximum and limits and where the function increases versus decreases and when the function is positive versus negative, it should help them when being introduced to higher order polynomials.
2.4.2 Factorisation of quadratic expressions

To factorise an expression of the form \( ax^2 + bx + c \) find the product of the first and last terms which is \( acx^2 \). Find two terms whose product is \( acx^2 \) and sum is \( bx \) to replace \( bx \). Factorise the resulting 4 terms by grouping. Mental structures required to perform the factorization is the ability to multiply terms correctly and simplify directed numbers correctly. Factorisation should be understood as the reverse of expansion since the resulting terms must expand to the original expression.

2.4.3 Solving quadratic equations

Quadratic equations may be set in the form \( ax^2 + bx + c = 0 \). To solve such an equation it can be by factorization, completing the square, the quadratic formula and the graphical approach. The equation has 2 roots which may be distinct and real if \( b^2 - 4ac > 0 \) and repeated if \( b^2 - 4ac = 0 \) and imaginary or complex if \( b^2 - 4ac < 0 \) but at ordinary level learners do not encounter complex roots.

When solving by factorization the expression \( ax^2 + bx + c = 0 \) is set in the form \((hx + x_1)(kx + x_2) = 0\) where the zero multiplier effect leads to \( hx + x_1 = 0 \) and \( kx + x_2 = 0 \) where the solutions of the equation are \( x = \frac{-x_1}{h} \) and \( x = \frac{-x_2}{k} \).

Factorisation is the easiest when the terms can be factorized easily but if they cannot be factorized with integer coefficients then completing the square or use of the quadratic formula has to be carried out. Completing the square leads to the quadratic formula. To solve \( ax^2 + bx + c = 0 \) by completing the square:

1. Make the coefficient of \( x^2 \) unity (i.e. 1) by dividing throughout the equation by the coefficient of \( x^2 \).
2. Take the constant term to the right hand side leaving the terms in \( x^2 \) and \( x \) on the left.
3. Complete the square by adding the square of half the coefficient of $x$ to both sides. The relation $(x+a)^2 = x^2 + 2ax + a^2$ has to be understood to see why half of the coefficient of $x$ is added.

4. After adding the square of half of the coefficient of $x$ the expression with $x^2$, $x$ and the square of half of the coefficient of $x$ is factorized.

5. Take the square roots of both sides.

6. Solve the resulting linear equations for $x$.

2.4.4. The quadratic formula

If the method of completing the square is applied to the equation $ax^2 + bx + c = 0$, ‘the formula’ for solving quadratic equations is derived.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the formula the equation must be set to zero on one of the sides. Learners have to make the transformation of the equation to the form $ax^2 + bx + c = 0$. The value of $b^2 - 4ac$ determines the nature of the roots. To use the formula the learner must have the ability to
substitute correctly into the formula and perform the arithmetic calculations correctly. The formula is quoted and used without derivation.

Equations can also be expressed in the form \((ax + h)^2 = k\) the process of squaring has to be reversed by taking square roots both sides such that \((ax + h) = \pm \sqrt{k}\). The square root of \(k\) means the positive or the negative square root in this case and the linear equations \((ax + h) = \sqrt{k}\), 
\((ax + h) = -\sqrt{k}\) are solved for \(x\).

2.4.5 The graph of a quadratic function

Figure 1: Graphs of quadratic functions

\[ f(x) = x^2 + 6x + 5 \]

\[ g(x) = -x^2 + 4x - 3 \]
The graph of a quadratic function is a curve called a parabola. When using the graph learners must interpret the graph and read points from the graph such as the $x$-intercepts which are solutions of the equation $ax^2 + bx + c = 0$ or the $x$-coordinates of the intersection of the graph of $y = ax^2 + bx + c$ and the line $y = l$ which are solutions of the equation $ax^2 + bx + c = l$. The equation $ax^2 + bx + c = l$ can also be set as $ax^2 + bx + g = 0$ where $g = c - l$ and learners must be able to figure out the processes that resulted in the expression $ax^2 + bx + g = 0$ and transform the equation back to $ax^2 + bx + c = l$ so that they can use the graph to solve the equation.

The graph can be used to find the maximum or minimum points and the line of symmetry and the graphical approach is used at ordinary level and is also emphasized in ordinary level mathematics textbooks.

### 2.5 Related Studies

Mu’awiya (2015) carried out a study with 126 senior secondary mathematics students randomly selected from 3 schools in Zaria a district in Nigeria and the purpose of the study was to analyse their problem-solving difficulties in quadratic equations. The study examined the problem-solving abilities of students of different academic abilities using the Jackson (1975) and Ashmore (1979) model. A descriptive survey design was used. Three types of instruments were applied which are the Mathematics Achievement Test (MAT), Mathematics Competence Test (MCT) and Problem-Solving Test in Quadratic Equation (PSTQ). The MAT was 25 multiple-choice objective test in algebra used to test students’ background in manipulating equations and other related topics. The MCT was also a multiple choice objective test in mathematics topics related to quadratic equations. The MAT and MCT were used to test students’ competency in mathematics and categorize them as high achievers and low achievers. The problem-solving test in quadratic equations (PSTQ) consisted of 5 numerical problems in quadratic equations designed to assess students’ problem-solving strategies with respect to their abilities and to identify the type of difficulties encountered by the students when scored within the framework of the Jackson (1975) and Ashmore (1979) model.
Mu’awiya (2015) observed that students had difficulties in defining a problem goal, recall of appropriate information and also with the correct strategies required to reason through and solve the problem. They recommended that the teacher as a facilitator should encourage the learners to concentrate on one point at a time and proceed stepwise in a logical manner to reduce attendant difficulty and improve critical reasoning. Difficulties students face in problem–solving depends on a number of factors such as failure to direct attention to appropriate information on problem statement, cognitive ability or ability to recall relevant mathematical knowledge from memory, students socio-economic background and outright lack of knowledge as well as method of instruction used (Henbee, 1992; Mittag & Taylor, 2001; Omwrirhiren, 2004; Zheug, 2007; Picciot, 2008 and Zakaria, 2001 cited in Mu’awiya).

Studies conducted by Polya (1957) and Mayer (2003) noted that learning how to solve problems in mathematics is to know what to look for. Mathematics problems often require established procedures and knowing what and when to apply them (cited in Mu’awiya, 2015). Buchanan (1987) noted that in identifying procedures a problem solver have to be familiar with the problem situation and be able to collect appropriate information, identify a strategy or strategies and use the strategy appropriately (cited in Mu’awiya).

Makonye and Nhlanhla (2014) investigated the errors learners show when they solve quadratic equations through factorization and they made use of the constructivist perspective of learning to explain learners’ errors and misconceptions. Research took place at a high school in the East Rand of Gauteng Province in South Africa. Most students’ errors arose from factorization. Learners were found to hold on to simple equations schema which they unsuccessfully used to assimilate solutions to quadratic equations when restructuring their schema was the only viable pathway. According to Erlwanger (1975) errors and misconceptions are caused by learners creating rules of their own that only works for them (cited in Makonye and Nhlanhla).

In the study by Makonye and Nhlanhla (2014) they observed that learners failed to come up with the factorization cognitive structure needed for the correct execution of factorization. There was failure to factorise correctly, catering mainly to satisfy two terms instead of all three so that on expansion it will lead to the original expression.
In a study of misconceptions about quadratic functions by Ibeawuchi and Ngoepe (2012) with a sample of 170 students from 17 classes in South Africa the following were some of the misconceptions identified:

1. Treating two different functions as equivalent for example the functions \( f(x) = x^2 - 3x - 4 \) and \( f(x) = 3x^2 - 9x - 12 \) and the reason given by the learners who had the misconception is that after dividing by 3 the second function it is the same as the first and other similar functions which are multiples of the first were treated as being identical. However, two quadratic functions which have different values in their leading coefficients are different as they differ in all other coordinates except the \( x \)-intercepts and the \( x \)-coordinate of the vertex. The learners treated the functions as if they were treating quadratic equations. The misconceptions emanated because the learners had completed solving quadratic equations and in this concept they were taught that equivalent equations are the same and hence they could operate on the simpler equations.

2. Limiting the graph of the quadratic function only to the visible region. There was tendency to read the graph of a quadratic function like a picture and therefore failure to conjecture the graph crossing the \( y \)-axis if the graph shown does not show the \( y \)-intercept. It was failure to interpret the quadratic function having an infinite domain. Similar findings were observed by Leinhardt, et al (1990) and Zaslavsky (1997).

3. Ascribing linearity to the quadratic function. For the students the midpoint of two close points that are on the parabola is in a straight line with the two points. For the learners the distance between the two points on the parabola is a straight line however it is a curve, since a parabola can pass through three collinear points no matter how close they may seem to be. Ascribing linearity also manifested itself in other forms including learners’ tendency of using straight lines instead of curves to join the points they plotted when drawing a parabola.

Over-attachment stems from the fact that it is linear functions learners are introduced to first and they over-generalise the conjectures they make when they learn the quadratic functions. Similar results were also observed by Zaslavsky (1997) and Dreyfus and Einserberg (1983). Ibeawuchi and Ngoepe (2012) also pointed out that each curriculum sequence has its own attendant misconceptions.
4. Describing a special point by only one coordinate as they described the vertex by its $x$-coordinate only whilst in fact it is a point and must be described by using a pair of coordinate axes.

Didis and Erbas (2015) investigated performance of 217 students in Turkey in solving quadratic equations with one unknown using symbolic equations and word problem representations. Data was collected through an open-ended questionnaire and semi-structured interviews. Students’ written responses and interview data were qualitatively analyzed to determine the nature of students’ difficulties in formulating and solving quadratic equations. The findings revealed that students performed better in symbolic equations compared with word problems. Arithmetic and algebraic manipulation errors were the main causes of difficulties in solving symbolic problems. In the word problems students had difficulties comprehending the context and were unable to formulate the equation to be solved.

Vaiyavutjamai and Clements (2006) studied the effects of classroom instruction on students’ abilities to solve quadratic equations. They attributed students’ difficulties from lack of both instrumental and relational understanding of solving quadratic equations. They suggested that relational understanding was low because of a teacher-centred instruction with a strong emphasis placed on the manipulation of symbols that increases instrumental understanding rather than the meaning of the symbols. Lack of relational understanding was suggested as the cause of misconceptions.

Kotsoupolos (2007) explained students’ difficulties in factoring quadratic expressions of the form $ax^2 + bx + c$ where $a \neq 1$ due to failure to recall main multiplication facts. The challenge is attributed to failure to find factors rapidly. The complication when the leading coefficient or the constant term has many factors was also studied by Nandakumar (2005) as cited in Didis and Erbas (2015).

Lima (2008) and Tall et al (2014) as cited in Didis and Erbas (2015) reported in their studies that students attempted to transform quadratic equations into linear equations and students showed tendency to use the quadratic formula as the only valid method in solving quadratic equations.

Makgakga (n.d) conducted a study in 5 South African schools in the Limpopo Province to investigate learners’ errors in solving quadratic equations by completing the square. Aim was to
diagnose errors learners made in solving quadratic equations by completing the square and explain the reasons why those errors were made. Errors and misconceptions were ascribed to teaching approaches used by teachers as the learners were not given an opportunity to discuss the concepts and they did not have enough time to practice their work. It also emerged that learners in the study were afraid to be mocked by their peers during the lessons and the learners themselves were not committed to their school work and some had parental role to play at their homes. Some of the teachers were not committed to their work as they often came to lessons late. Affective factors and not merely cognitive factors were also found to affect performance.

This study will add to the body of knowledge about errors and misconceptions in solving quadratic equations. Further clarification on the origins of errors and misconceptions is being sought for example the origin of the short formula will be sought.

2.6 Conclusion

In this chapter the theoretical framework that encompasses constructivism, procedural and conceptual knowledge, concept definition and concept image is discussed. The theories are explained and discussed with how they are linked to misconceptions. Related studies and their findings are discussed and the next chapter focuses on the research methodology, research design, data collection and the instruments used.
CHAPTER 3
RESEARCH METHODOLOGY

3.1 Introduction
In this chapter the research procedures that were used to investigate the research question are described and this includes the research design, the sampling procedures for the identification of the participating schools and students, the research instruments and their validation, the data gathering and data analysis processes.

3.2 Research design and method
A qualitative research approach was used in the investigation. According to Denzin and Lincoln (1994) as cited in Mhlanga and Ncube (2003) a qualitative research study things in their natural settings attempting to make sense of, or interpret phenomena in terms of meanings people make of them. It emphasizes meanings, experiences and descriptions. Qualitative data consists of gathering information during research that has not been quantified in any rigorous way. Data generated is measured non-numerically although often placed into categories which follow an appropriate order for example answers to a problem can be placed in one of a number of descriptive categories.

A qualitative approach was also applied by Childers and Vidakovic (2012) whose interest was on students’ understanding of the concept of vertex of quadratic functions. Nielsen (2015) also used a qualitative approach to investigate students’ mathematical thinking and understanding of quadratic functions. The qualitative approach is fitting for an endeavor that is to describe a phenomenon in the social world from the standpoint of the individuals who are part of the ongoing event which cannot be quantified numerically. Students’ errors and misconceptions cannot be measured numerically but they can be categorized and described so for this study a qualitative approach is suitable. A qualitative method is useful to probe deep within the minds and for attitudes, feelings and reactions of the respondents. In this research students were expected to display their conceptualization of solving quadratic equations and the qualitative research approach enables one to answer the research questions. A qualitative research approach in education according to Mason (2006) enables the researcher to understand and explore the students’ thinking and conceptions.
Data was collected regarding students’ misconceptions and errors in solving problems on quadratic equations. In this study the participants were 249 form 3 mathematics students. Students’ approach was analysed through a test item with 7 open ended questions and a semi-structured interview followed up on 31 students who were selected based on their written responses. A semi-structured interview approach allows a follow up on emerging trends in detail and allows perspectives to emerge freely (Vulliary et al, 1990)

The research design followed a case study approach. According to Mhlanga and Ncube (2003) a case study is an enquiry in which the researcher confines his/her effort to a limited area and focus with intensity on some phenomena of interest. A case study approach allowed the researcher to gain insight into the misconceptions and errors the learners had as they solved problems on quadratic equations. The advantage of a case study method is that it allows the researcher to focus and gain insight on a specific phenomenon but the disadvantage is that the findings cannot be generalized.

3.3 Research site and population

The research was carried out at three 3 schools in Mashonaland West Province of Zimbabwe. A population constitutes all the characters under investigation and all the individuals that are considered in the project (Mhlanga & Ncube, 2003). The population of the study comprised of all form 3 mathematics students from the 3 schools. The schools code named MC, KD and HW are in the same district to reduce transport costs. MC has 188 form 3 mathematics students, KD has 197 and HW has 92 students. MC has 4 mathematics classes which were being taught by 2 teachers and KD has also 4 mathematics classes taught by 1 teacher and HW has 2 classes taught by 1 teacher. The district has a total of 42 secondary schools.

At MC the 4 classes have been streamed according to ability with 3D having high performing students followed by 3C, 3B and 3A with the low performing students. At KD and HW the classes were also streamed.

The researcher teaches at one of the school but not the form 3 classes to reduce bias. In this way there was no conflict between researcher and teacher. At all the schools they had covered the topic at most 3 months from the data collection time.
3.4 Sampling procedure and sample

Best and Kahn (1993) described a sample as a portion of the population selected for observation and analysis. It is usually difficult to study and work with the whole population so a sample is selected to represent the population and make inference about the large population from the sample. A sample is part of the target population under study.

Convenient sampling was used to select participants in the study and they were volunteers. According to Mhlanga and Ncube (2003) convenience sampling is made up of those participants that are conveniently available or those that the researcher can easily stumble on and it also makes wide use of volunteer subjects. Administrative costs and time limitations often force researchers to adopt this sampling procedure. It is suitable where very low levels of generalization are to be made as it can produce highly biased samples.

Table 3.1 Distribution of the sample

<table>
<thead>
<tr>
<th>SCHOOL</th>
<th>NO. OF PARTICIPANTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>137</td>
</tr>
<tr>
<td>KD</td>
<td>62</td>
</tr>
<tr>
<td>HW</td>
<td>50</td>
</tr>
<tr>
<td>TOTAL</td>
<td>249</td>
</tr>
</tbody>
</table>

3.5 Data collection instruments

Data was collected through a written test and one-on-one interviews with the selected students. Vaiyavutjamai, Ellerton and Clements (2006), Didis and Erbas (2015), Zaslavsky (1997) used a written test to collect data on students’ understanding of quadratic functions and quadratic equations. Vaiyavutjamai et al also used interviews to solicit how students approached the test questions.

The test item had 7 questions on quadratic equations and functions which were open ended and participants were to solve them individually and show working. Solutions were then analysed. It consisted of standard and non-standard problems. Test items that were familiar were also chosen so that participants would be able to answer. Questions were adapted from Vaiyavutjamai, Ellerton and Clements (2006) and two standard textbooks used at form 3, The New General
Mathematics Books and Focus on Mathematics Book 3. The test items provided an opportunity for qualitative analysis and reasoning.

The test items were designed with the following considerations:

- Inclusion of non-standard problems or procedures to minimize students’ reliance on rote memorization of algorithms. Some of the problems were not familiar so that the students would not likely use routine solutions based on readily available algorithms and thus would have to reason out the solution. Non-familiar problems would prove helpful in detecting students’ difficulties as well as their strengths with respect to solving problems on quadratic equations. Inclusion of non-familiar items was also used by Zaslavsky (1997).
- Use of scientific (non-programmable or graphical) was encouraged to reduce calculation errors (Zaslavsky, 1997).
- To allow analysis of methods used, students were encouraged to show their working.
- Problems were also selected that are familiar so that most have an opportunity to answer. If the test becomes too difficult students will not answer.

According to Schultze and Avital (2010) interviews are attempts to understand the participants’ point of view. It consists of the researcher asking the respondents a series of questions and in this research face to face interviews were conducted. Its’ advantage is that the researcher is able to get information that could not be obtained from a written solution and in this research the interviewer solicited for justification of the participants’ work, further exploration and explanations of why they chose certain strategies or gave certain solutions. Another advantage is that follow up questions can be made if the respondent misinterprets the original question.

Interviews were effective as they enabled the researcher to obtain justification of respondents’ written solutions. The researcher conducted the interviews himself with participants at MC.

3.5.1 Validation of instruments

Validity is concerned with the soundness or effectiveness of a measuring instrument (Leedy, 1993). The instrument should measure what it is supposed to measure and questions have to ask about what the instrument is trying to measure. Data is valid if it provides a true measurement,
description or explanation of what is being measured. All questions in the test item and the interviews ask information on quadratic equations.

To ensure that the test items and the interview questions have content and face validity, 3 teachers at MC and the teacher at KD were requested to scrutinize the instruments to establish the validity of both the content and the format of each research instrument. Discussion of the intentions of the questions ensued and an agreement was made on the inclusion or exclusion of certain questions.

Teachers were also requested to give possible errors that could be displayed by students. A pilot study was done with 17 students at a school code named NG in a different district who completed the test items and the researcher conducted an interview on 5 students. The instruments were then modified and fine-tuned in consultation with the 4 teachers. Language related errors were noted on the test items and on the interview structure and changes were effected to improve their usefulness. At the end it was agreed that the knowledge required to solve each of the problems could be assumed to be possessed by their students and they could answer all questions with some exceptions.

At MC the teacher hinted that students could have challenges in answering question 1 as they had not done similar problems in class. The other teacher at MC who is taking 3A and 3B said the pupils were capable of answering all questions unless they have forgotten the material. He hinted that the students would answer correctly if given a test immediately after a topic has been learnt in class. At HW the teacher said question 6 will pose challenges as word problems are generally difficult for them as they fail to interpret questions. At KD the teacher said the students were capable of answering all questions and none could pose difficulties.

The interview guide questions were refined with input from the 4 teachers who assisted with wording, sequencing and content to be questioned.

Two teachers at MC who validated the instrument and gave their opinions have each over 20 years teaching experience and one is an examiner at ordinary level with the local examination board and they assisted with wording and question sequencing. The other is an examiner at advanced level. The third teacher at MC has 4 years teaching experience. At KD the teacher has 8 years teaching experience and is an examiner at advanced level. At HW the teacher has 6 years teaching experience in 2 subjects mathematics and economics.
3.5.2 Reliability

Reliability is the degree of consistency that an instrument or procedure demonstrates (Best and Kahn, 1995). The same results should be obtained if the same instrument was used under similar conditions. Best and Kahn further posits that data is reliable if other sources using the same investigative procedures will produce the same results.

In this study reliability was enhanced by reducing anxiety during the test and informing the participants that results will be published anonymously and will not be used as part of their common assessment. Anxiety was also reduced by the test items being administered by teachers whom the participants were familiar with. Purpose of the research was also explained to the participants. Reliability was also enhanced by choosing appropriate wording for the test items and interviews.

An interview guide was used which had open-ended and closed questions. In open ended questions the participants would give an explanation on their work to give more insight. All the respondents were asked the same initial questions and variations were due to the manner they would have responded on the test.

Validity and reliability achieved through triangulation. Triangulation involves using multiple data in an investigation to produce understanding and one can be more confident if different methods lead to the same result. Results from test items and interview compared for consistency to see if they are the same. Triangulation facilitates validation through cross verification from two or more sources (Cohen & Manion, 2000).

3.6 Preparation for the study

Preparation for data collection involved:

- Obtaining permission from the Mashonaland West Provincial Education Director.
- Obtaining permission from the heads of selected schools.
- Validation of research instruments.
- Identifying participants who volunteered. Participants could withdraw from the study at any time.
- Consulting and agreeing with the teachers at the 3 schools the data collection schedule.
The identity of participants was protected and codes were allocated to them. In the research instruments and in all data collected from the participants they are referred by their code names. This in line with the principle of research ethics that participants should be offered the opportunity to have their identity hidden in the research report (Oliver, 2003).

### 3.7 Administration of the main study

Before the data collection process, the researcher met with the heads of the concerned schools to discuss the whole research process and clarify any issues that they might have about the whole research process.

The time to administer the test items was agreed and at all 3 schools it was after lessons. Data was collected over a period of 2 weeks. At the school were the researcher is teaching, the test items were administered personally and interviews were conducted with participants from this school to reduce anxiety as the participants were acquainted with the researcher. At the two other schools the teachers teaching the classes administered the tests to reduce anxiety as the learners knew the teachers.

Instructions for completing the test items were discussed with the teachers and appropriate conduct explained such as there is no time limit to complete the test item as the research was not about finding out how participants quickly answer the test items, no formulae to be given and the teachers not to explain the test items to the participants.

After the test items were administered a preliminary analysis was done for tentative themes and participants at MC who were identified to be of interest were requested to participate in one-on-one interviews. Test items took on average 50 minutes to be completed by participants and the one-on-one interview took on average 15 minutes. The interviews were audio-taped and notes were also written of the interview process.

### 3.8 Data analysis

Data was analysed from the test items and coding for each question was done according to the emerging categories. Initial codes were formulated and other codes emerged as data processing progressed. Data analysis was ongoing and began at the onset of my first interview. I engaged in
preliminary analysis as I interviewed students which informed my decision about potential question order as well as which follow-up questions to ask each individual.

For interviews data analysis was for triangulation with the written test and to probe further and get explanations from the participants on their conceptualization of the test items. Data was analysed using the theoretical framework of constructivism, the concept definition and concept image. Analysis was conducted question by question to get the frequency from each school which answered with a particular strategy or gave a particular solution. Common errors and misconceptions from the three schools were tabulated.

3.8.1 Data analysis phases

The first phase was coding of each students’ test item transcript based on the approach to the research questions. Coding was developed from the pilot tests and interviews, conceptual framework and existing literature. More codes were added as data analysis progressed. Themes were identified in students solving of quadratic equations. Patterns, errors and misconceptions in solving equations were identified.

In the second phase of data analysis interviews were used and focus was on obtaining students justification of their work and to identify if an error was just a slip the student had made in the test item. Logical basis of their solutions was also obtained. Records were kept in a notebook with students code names. If an error occurred at least two times from a school and was justified or its logical basis explained in an interview then it was regarded to constitute a misconception and a similar analysis is found in literature by McIntyre (2007). Misconceptions are also identified as they are held by many people and in this research misconceptions are also identified if errors are held by participants at the three schools (Li, 2006).

3.9 Ethical considerations

Researchers need to keep in mind the human aspect in their research and thus ethics play a pivotal role in studies to protect participants’ right to life, respect and dignity. Resnik (2015) defined ethics as norms for conduct that distinguish between acceptable and unacceptable behaviour. Ethics in research is concerned with the right and wrong to follow in conducting the
research (Mhlanga & Ncube, 2003). Ethical considerations have to be taken into account to ensure objectivity and integrity. According to Schurink, Schurink and Poggenpoel (1998) important ethical issues to be included are:

- Voluntary participation to be part of the data gathering process. Participants are also to be informed that they can voluntarily leave the project at any time without any consequences on their part.
- Informed consent need to be obtained from the participants and their superiors. Participants will not have to lose learning time at school by participating in the project and data collection was done after lessons. Written consent was obtained from the Mashonaland West Provincial Education Director.
- Confidentiality and anonymity should be assured in the research and when results are published. For this study participants were code named.
- Feedback regarding the results was contractually agreed with the provincial education office.
- Interview items, note books, audio tapes, test item scripts and all observation notes were kept in a locked and safe place to ensure that no one other than the researcher could assess the information.
- The principle of beneficence was communicated to the participants. They were informed of the potential benefits of the research study which is to improve the teaching and learning of mathematics. Participants would also gain further proficiency in tackling problems on quadratic equations.

3.10 Chapter conclusion

In this chapter the research methodology was detailed with regard to the research procedure, the research site, the population size, the sample, the data collection instruments, validation of the research instruments and data collection processes. Attention was drawn on the validity and reliability of instruments and ethical considerations.
CHAPTER 4
DATA PRESENTATION, ANALYSIS AND DISCUSSION

4.1 Introduction

In this chapter the main findings are presented and analysed. This will involve a description of the errors and misconceptions participants showed in solving quadratic equations. A qualitative analysis will involve findings from the test items and interviews. Analysis will follow question by question of the test items. Possible reasons for the challenges, errors and misconceptions will be explained. Although not the focus of the research I explored how high achieving and low achieving students differed on the types of challenges errors and misconceptions.

4.2 Results from the test

4.2.1 Question 1

To learn how students understand the concept of root of an equation I asked them the question: If one of the roots of the equation \( x^2 - ax + 21 = 0 \) is 3, what is the value of \( a \)?

Students’ responses were categorized as shown in the table below

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solved correctly and with complete correct solution</td>
<td>MC 78</td>
<td>135</td>
</tr>
<tr>
<td></td>
<td>HW 23</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 34</td>
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</tr>
<tr>
<td>Substituted ( x ) by 3 but failed to simplify and get correct solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Failed to solve resulting linear equation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Attempted to solve as an equation</td>
<td>MC 16</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>HW 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 2</td>
<td></td>
</tr>
<tr>
<td>Correct answer but wrong working</td>
<td>MC 1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>HW 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 0</td>
<td></td>
</tr>
<tr>
<td>Substituting ( a ) by 3</td>
<td>MC 0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>HW 2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 0</td>
<td></td>
</tr>
<tr>
<td>Attempt difficult to follow and abandoned</td>
<td>MC 10</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>HW 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 6</td>
<td></td>
</tr>
<tr>
<td>( x^2 ) replaced by ( 3^3 )</td>
<td>MC 1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>HW 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 0</td>
<td></td>
</tr>
<tr>
<td>No solution</td>
<td>MC 25</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>HW 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>KD 12</td>
<td></td>
</tr>
</tbody>
</table>

Example of learners’ work showing the errors and misconceptions above is given and a discussion on their sources is made.
An error noted is an attempt to solve for the value of $x$ in the equation and learners did not use the value of $x$ given or make a substitution of $x$ by 3. Learners had to deduce that one of the possible values of $x$ is 3 and make a substitution. It emanated from failure to understand the concept of root of an equation. Attempts to solve as an equation had some using the quadratic formula or factorization to solve the equation for $x$. Even for those who attempted solving as an equation, the expression for the root from the quadratic formula had to be equated to 3 if there is a clear understanding of the root of an equation. The understanding of the concept of root of an equation was lacking. The error was exhibited in all 3 schools but with a high frequency in the low performing classes at MC.
Table 4.2 *Categories of responses from achieving and high achieving learners at MC*

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>Low achieving learners</th>
<th>High achieving learners</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3A</td>
<td>3B</td>
<td>3C</td>
</tr>
<tr>
<td>Solved correctly and with complete correct solution</td>
<td>5</td>
<td>14</td>
<td>29</td>
</tr>
<tr>
<td>Substituted $x$ by 3 but failed to simplify and get correct solution</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Failed to solve resulting linear equation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Attempted to solve as an equation</td>
<td>7</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Correct answer but wrong working</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Attempt difficult to follow and abandoned</td>
<td>6</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$x^2$ replaced by $3^3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>No solution</td>
<td>7</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

Even those who attempted as if solving for the unknown had the wrong quadratic formula.
From the interviews some of the definitions of root of an equation given by respondents were that: “a root is one you get in the exam after working to find $x$ or $y$”. This definition shows an action understanding of the root and not an object view. An action has to be performed to find the root and there is need for external stimuli. 3D1 has an object view of the root as he said, “a root is the value that makes the statement correct”. Even some learners who solved the question correctly defined a root as, “the result of the equation” and after probing he said, “After solving what we get as the answer is the root”. They had a concept image of the root of an equation that did not differ from the concept definition but the challenge was in application. The question was unfamiliar to respondents and some displayed failure to accommodate the stimuli with what they already knew.
There was also interpretation of roots of an equation to mean a factor as defined by 3D16 who could not answer question 1 correctly as he said, “roots are such that they give us 21 on multiplying and –a on adding”. Thus some learners had a flawed concept image of the root of an equation and leading to errors and misconceptions in application of the concept of root of an equation. Learners need to have the correct concept image of the root of an equation and its associated schemas so as to be able to manipulate on the root of an equation.

**Wrong quadratic formula**

An error noted which permeates some learners work is the use of wrong quadratic formula.

**Figure 4: 3A37 solution to question 1 showing wrong quadratic formula**

There was justification by some learners of the use of the short formula displaying a misconception. For 3C31 the formulae \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) and \( -b \pm \sqrt{b^2 - 4ac} \) do not differ. Some learners will divide at the end like what was done by HW41 and KD15 though they have the short form at the start. Other respondents like 3A24 and 3B43 fail to make that “recovery”.

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Algebraic malrules

Algebraic malrules were also used like the one shown below in a students’ work

Figure 5: 3A8 response to question 1 showing the category of algebraic malrules
A flawed concept image of a fraction caused the respondent to simplify $\frac{x^2 - ax}{x^2}$ as $-ax$. The schema formed of fractions and division led to the misconception that division of equal terms will result in zero as seen from $\frac{x^2}{x^2}$. The learner had wrongly assimilated the concept of division as he had encountered it in terms such as $3x$ which on division by 3 gives $x$, so on dividing $x^2 - ax$ by $x^2$ there was overgeneralization of the concept of division of a products with one of the multipliers. This was also use of a procedure without clear understanding.

Figure 6: 3A13 solution to question 1 showing algebraic malrules

Malrule used by 3A8 was also observed on his question 4 and this demonstrated the misconception in simplification of algebraic fractions. Misconceptions in dealing with algebraic terms involve addition of unlike terms was also done by KD60. There is also addition of powers
of $x$ which only works for multiplication of exponentials with the same base. Rules used in division of algebraic fractions only work for terms being multiplied and learners have over-generalised them creating a misconception. Misconceptions arise from a valid rule being applied in a situation where it is not applicable.

Figure 7: HW45 response to question 1 showing a misconception

![HW45 RESPONSE TO QUESTION 1](image)

Figure 8: HW45 response to question 2 showing the misconception also reflected in question 1

![SOLUTION](image)
HW45’s error is systematic and is being consistently applied. An error is labelled systematic when, there is a repeatedly occurring incorrect response that is evident in a specific algorithmic computation, Cox (1975). Systematic errors are labelled as misconceptions. There is subtraction of unlike terms and the student treated them as like terms. In learning there was encounter with like terms first and there was failure to accommodate unlike terms. There is need for new mental structures to deal with unlike terms. The learner did not realize the conflict that arises when treating unlike terms as like terms. There was no attempt to form new mental structures, hence assimilating incorrectly the unlike terms as like terms.

**Failure to understand that \( a \) is constant**

Failure to interpret the question as if meaning that they have to solve the equation was shown by HW36 who obtained two values of \( a \), -10 and 10 and could not understand that \( a \) is a constant.

Failure to interpret the question is also demonstrated by HW45 who substituted \( a \) by 3. For such students their understanding of the structure of a quadratic equation is superficial.

**Failure to solve resulting linear equation**

After substituting \( x \) by 3 some could not solve correctly the resulting linear equation demonstrating that the solution of linear equations is a challenge for some. Working with sign of terms was a challenge as shown in HW24’s and KD27’s work.
After the correct substitution of \( x \) by 3, the respondent could not solve the resulting linear equation and there is confusion with \((-a) 3\) as it was erroneously converted to \(-a + 3\). There is evidence of errors as \( a \) was converted to 9 and \(-a \times 3\) to \(-a + 3\).
Errors evident in the respondent’s work are in dealing with directed numbers as 9 should be positive.
Imposing $\sqrt{-x}$ as $\sqrt{x}$

Figure 12: 3B34 response to question 1 showing imposing $\sqrt{-x}$ as $\sqrt{x}$

\[ \begin{align*}
  x^2 - ax + 21 &= 0 \\
  a &= 1, \quad b = 1, \quad c = 21
\end{align*} \]

\[ \begin{align*}
  -b \pm \sqrt{b^2 - 4ac} \\
  2a
\end{align*} \]

\[ \begin{align*}
  -1 \pm \sqrt{1 - 25} \\
  -1 \pm \sqrt{24} \\
  -1 \pm \sqrt{12} \\
  -1 \pm \sqrt{12} \\
  -1 \pm 3.46 \\
  -1 \pm 3.46
\end{align*} \]

$3B34$ RESPONSE TO QUESTION 1
Figure 13: 3B34 response to question 3 showing imposing $\sqrt{-x}$ as $\sqrt{x}$

The learner imposed square roots of negative numbers to be equal to square roots of positive numbers to get over the hurdle and this shows creation or imposition of incorrect procedures by the learner. The learner could not realize the conflict showing use of procedures without understanding. The square root of a negative number was operated as that of a positive number. Arithmetic errors and failure to substitute into the quadratic formula is also evident. The participant did not also check his work to deal with the conflict that arose.
4.2.2 Question 2

To understand learners’ proficiency in solving quadratic equations the following question was posed:

Solve the equation \(4x^2 - 12x + 5 = 0\).

Category of responses, errors and misconceptions are tabulated below.

Table 4.3 Categories of responses from question 2

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>HW</td>
</tr>
<tr>
<td>Solved correctly by factorisation</td>
<td>64</td>
<td>0</td>
</tr>
<tr>
<td>Failed to factorise correctly</td>
<td>19</td>
<td>18</td>
</tr>
<tr>
<td>Used quadratic formula correctly</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>Attempt to isolate (x) as if solving a linear equation</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Wrong quadratic formula and substitution</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Procedure not clear</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Short form of quadratic formula</td>
<td>19</td>
<td>6</td>
</tr>
<tr>
<td>Imposing (\sqrt{-x}) as (\sqrt{x})</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Wrong value of (-b)</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>No solution</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

**Failure to factorise correctly**

A procedural error was shown by -12x being replaced by inappropriate and wrong terms for example \(-x\) and \(-11x\) or \(-3x\) and \(-9x\). A typical example is solution by HW9.
The learners who failed to factorise demonstrated the shortfall of using procedures without understanding. There is need for learners to have conceptual and procedural knowledge of factorization.

The learner did not check whether if when the factored form is expanded the result is the original expression. There was consideration of only the quadratic term. HW22 focused only on the first and last terms.
Figure 16: HW22 response to question 2 showing failure to factorise

The findings agree with those by Kotsopoulos (2007) that factorization can be tricky if the leading coefficient is not 1.

Wrong quadratic formula

The short form of the quadratic formula was an error that permeated some learner’s work. It seems for some the expressions $-b \pm \sqrt{b^2 - 4ac}$ and $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are equivalent. 3C48 on being asked in an interview how $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $\frac{\pm b \sqrt{b^2 - 4ac}}{2a}$ differs he said after some consideration that they are the same.

3C31 consistently gave $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ as the formula and said brackets are the only difference with the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ in the interview.
Some learners will divide \(-b \pm \sqrt{b^2 - 4ac}\) by \(2a\) at the end like HW41 and KD15 though they have the incorrect short formula at the start and some like 3A24 and 3B43 fail to make that “recovery”.

Figure 17: KD15 response to question 2 showing inconsistent division by \(2a\)

Figure 18: 3A24 response to question 2 showing the use of the short division
Some learners know that there is $\pm b$ but do not know where to place it in the formula. The use of the short formula is a misconception identified, some learners knew that they have to divide by $2a$ but could not place it correctly in the formula.

Short division problem could emanate from the large numerator. Working with large numbers increases chances of making a mistake (Donald, 2007). It could also emanate from failure to work with fractions. A participant in an interview said that the short division form is easier to
work with and erroneously also made the simplification $\frac{\sqrt{8}}{2} = \frac{\sqrt{2^3}}{2} = \frac{\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}}{2} = \sqrt{4} = 2$ when asked to solve an equation to further explore how he deals with fractions.

**Failure to substitute into the equation correctly**

Figure 21: *HW29 response to question 2 showing failure to substitute correctly in the formula*

Wrong value of $-b$ is given in this case and there is also failure to simplify $12^2 - 4 \times 4 \times 5$ correctly. Failure to understand the structure of the quadratic equation results in failure to substitute into the equation correctly.
Imposing $\sqrt{-x}$ as $\sqrt{x}$

This is evident of learners devising ways which are incorrect to solve a problem if they have reached an impasse. In their attempt to reach an equilibrium and accommodate the unfamiliar experience, learners create incorrect logico-mathematical structures resulting in malrules. At form 3 and with the problems given they will not encounter and need not deal with the square root of a negative number. Since the learners have meaning to the misconceptions they are reluctant to give them up even in the face of having reached a dead end as explained by Smith, DiSessa and Roschelle (1993).

**Linearising the equation and mis-application of the zero-product property**

Figure 22: 3C19 response to question 2 showing over-generalisation of the zero product property

![Figure 22: 3C19 response to question 2 showing over-generalisation of the zero product property](image)

Figure 23: 3C52 response to question 2 showing the attempt to linearise the equation

![Figure 23: 3C52 response to question 2 showing the attempt to linearise the equation](image)
The quadratic equation was being solved with the techniques they have learnt in solving linear equations. The strategy used by the learner is undoing so as to isolate $x$ resulting in dividing by the unknown term on both sides. There is need to adapt their schema so that they have the schema about quadratic equations and see the futility of the schema of linear equations.
Algebraic malrules

A challenge for some learners is simplification of algebraic terms.

Figure 25: 3A38 response to question 4 showing algebraic malrules

\[ x^2 - 2x = 4 \]
\[ x^2 = 4 + 2x \]
\[ \frac{x^2}{x} = \frac{6x}{x} \]
\[ x = 6 \]

4 + 2x is erroneously simplified to 6x. There is overgeneralization through assimilation and the operating on unlike terms as if they are like terms. The learner had encountered the addition of like terms earlier but there is need for accommodation to deal with unlike terms.
The addition of powers is applicable in multiplication of indices with the same base. The rule is not applicable for addition of indices even those with the same base. The learner had invented the rule because he/she had encountered a similar case it is applicable and this caused the misconception when used in a situation it is not applicable.
For questions 3 to 5 some of the errors and misconceptions exhibited are the same as in questions 1 and 2 and so will not be repeated.

4.2.3 Question 3

Participants were asked to solve the equation $7m^2 = 3m$. Categories of responses are tabulated below.

Table 4.4 Categories of responses from question 3

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>HW</td>
</tr>
<tr>
<td>2 correct roots obtained by factorization</td>
<td>41</td>
<td>19</td>
</tr>
<tr>
<td>Failure to factorise</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>Only one solution $m = \frac{3}{7}$ was obtained due to division by the</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>unknown value $m$ or solution may not be $m = \frac{3}{7}$ but division by</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m$ was attempted</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Only one solution $m = 0$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Factorised correctly to $m(7m – 3)$ but failed to deduce correct</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>solutions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Correct solutions obtained by quadratic formula</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Quadratic formula used incorrectly</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Taking square roots both sides or one side</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>Short form of quadratic formula</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Work difficult to follow for example $7m^2$ converted to $49m$ or $7m^3$</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>No solution</td>
<td>9</td>
<td>0</td>
</tr>
</tbody>
</table>

Errors and misconceptions observed through solving this question are:

Failure to factorise

This is seen largely in learners’ work at HW and hence their overreliance on the quadratic formula and this overreliance is also seen in question 2. The failure to factorise was high in question 2 as there were 45 failures compared to 20 in question 3. Learners should not rely on
one method to solve quadratic equations as there are alternative methods which are simpler in some given cases.

**Applying an incorrect procedure like taking square roots**

Figure 27: 3B16 response to question 3 showing taking square roots both sides to isolate x

![3B16 RESPONSE TO QUESTION 3]

Figure 28: 3A25 response to question 3 showing taking square roots both sides to isolate x

![3A25 RESPONSE TO QUESTION 3]

Taking square both sides was done to isolate the unknown and this was an overgeneralization of the method to solve linear equations. For 3B16 method used of taking square roots is appropriate for question 5 but was used with futility on question 3 and question 4.
Failure to solve linear equations correctly

Figure 29: 3B59 response to question 3 showing failure to apply zero-product property and solve resulting linear equations

Learners encounter linear equations at form 1, 2 and 3 but some have not developed the proficiency to solve them and this posed a challenge when solving quadratic equations. In this instance the participant could also not realise when to apply the zero product property.

Failure to understand the structure of the equation

A cursory glance could give one solution \( m = 0 \) but learners had the predisposition to apply algorithms and rules without understanding the structure of the equation. Failure to identify this solution can pose difficulties for learners in applying the zero-product property. Learners need to understand the concept of root of an equation as the value which when substituted into the equation results in both sides being equal.

Dividing both sides by \( m \) resulting in loss of solution

If two factors have a product of zero, then one or the other of the factors must be zero and failure to understand this property leads to division by the unknown term and loss of a solution. This result confirms to similar findings by Didis et al (2011) and Kotsopoulos (2007). Failure to
understand the zero product property is also seen in some learners’ work in question 4 where it is applied incorrectly.

4.2.4 Question 4

The participants were posed the question: Solve the equation \( x^2 - 2x = 4 \)

Table 4.5 *Categories of responses from question 4*

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>HW</td>
</tr>
<tr>
<td>Solved correctly by quadratic formula or obtained values equivalent to ( \frac{2 \pm \sqrt{20}}{2} )</td>
<td>45</td>
<td>5</td>
</tr>
<tr>
<td>Wrong quadratic formula</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Short division in the quadratic formula</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>Failure to substitute in formula correctly</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>Wrong value of (-b)</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>Solved correctly by completing the square</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Completing the square incorrectly as coefficient of leading term is not 1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Zero product misapplied LHS factorised and all terms equated to 4</td>
<td>11</td>
<td>26</td>
</tr>
<tr>
<td>Imposing ( \sqrt{-x} ) as ( \sqrt{x} )</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Attempt to factorise the expression</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>Attempt to isolate ( x ) and solve as a linear equation</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>No solution</td>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

Errors and misconceptions noted are:

**Over generalization of zero product property**

The result that if \( a \times b = c \) then either \( a = c \) or \( b = c \) works only when \( c = 0 \) and this is the property that has been over-generalised by the learners. This was a common misconception as seen in the learners’ work below.
Failure to understand the effect of multiplication by zero could also result in learners’ missing the solution \( m = 0 \) in question 3. Learners need to be aware and realise the effect of multiplication by zero and use it when solving equations. Over-generalisation of zero product property was also seen in studies conducted by Vaiyavutjamai and Ellerton (2004) and Didis and Erbas (2015).
Wrong values of $a$, $b$ and $c$ used in the quadratic formula

Figure 31: 3A44 response to question 4 showing failure to substitute into the quadratic formula correctly

Failure to substitute correctly into the formula was noted as wrong value of $-b$ was used. Another error is in subtraction as $4 - 16$ is not $12$. This indicates a misconception due to overgeneralization of the commutativity of addition which does not apply for subtraction.
Figure 32: 3A44 response to question 4 showing use of the wrong quadratic formula

The short form of the quadratic formula and the wrong value of $-b$ were used. The erroneous use of the short form of the quadratic formula was explained in detail under responses to question 2.

**Attempt to solve the equation as a linear equation**

Misconceptions in simplifying algebraic expressions and attempts to use techniques of solving linear equations were also noted. Examples of misconceptions in simplifying algebraic expressions were discussed under responses from question 1. Learners need to accommodate the quadratic equation schema to be successful in solving them and not hold onto the linear equation schema.
4.2.5 Question 5

The participants were posed the question: Solve the equation \((x+1)^2 = 25\)

Table 4.6 Categories of responses for question 5

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solved correctly and obtained two roots by taking square roots and (\sqrt{25}) is given as (\pm 5)</td>
<td>50</td>
<td>92</td>
</tr>
<tr>
<td>Expanded correctly and solved the resulting equation correctly</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>Wrong expansion ((x+1)^2 = x^2 + 1) or other form ((x+1)(x-1))</td>
<td>16</td>
<td>28</td>
</tr>
<tr>
<td>One root where only the positive square root of 25 was considered and hence obtained one root</td>
<td>40</td>
<td>50</td>
</tr>
<tr>
<td>At some point (\sqrt{25}) evaluated as (\sqrt{5})</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Failure to simplify from (x + 1 = \pm 5)</td>
<td>9</td>
<td>26</td>
</tr>
<tr>
<td>Zero product property overgeneralized resulting in the expression (x + 1 = 25)</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>Expanded correctly but incomplete solution</td>
<td>11</td>
<td>18</td>
</tr>
<tr>
<td>Work difficult to follow through and could not fit into the other categories</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>No solution</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

Errors and misconceptions of interest are:

Wrong expansion

\((x+1)^2\) set as \(x^2 + 1^2\) by some participants. Squaring is not distributive over addition. The misconception springs from the fact that \((xy)^2 = x^2y^2\) but \((x+y)^2 \neq x^2 + y^2\). 3A8 on being interviewed said, “\(x \times x = x^2, 1 \times 1 = 1\) so \((x+1)^2 = x^2 + 1\)”, so the learner has a misconception as there is a logical basis for the error. Cox (1975) labelled such errors as systematic as there is a repeatedly occurring incorrect response that is evident in a specific algorithmic computation and systematic errors are misconceptions. It is also important to note that if a learner could not expand, factorization the reverse of expansion will pose challenges as experienced.

Expanding \((x+1)^2\) as \((x+1)(x-1)\)

Reasons for such a simplification were not clear
**Failure to solve the resulting linear equations** \( x + 1 = \pm 5 \)

Simplification of directed numbers posed challenges for some participants. There is also failure to deal with \( \pm 5 \). The \( \pm \) caused confusion for learners implying that they had not formed the appropriate mental and cognitive structures to deal with it.

Figure 33: *HW47 response to question 5 showing failure to understand the \( \pm \) sign and solve linear equations*

There is also an error in taking \( \sqrt{(x+1)^2} \) as \( x - 1 \).

**Square root is not distributive**

Figure 34: *3D1 response to question 5 showing misapplication of the distributive property on square roots*
This is seen from the working on the left hand side and since this gives the same solution \( x = 4 \) the learner has some belief that \( \sqrt{(x+1)^2} = \sqrt{x^2} + \sqrt{1} \). Misconceptions result from a flawed procedure that may work in some instance but not in all general cases. The learner had created a malrule.

**Dealing with the \( \pm \)**

Figure 35: 3C32 response to question 5 showing failure to understand the \( \pm \) sign and solve linear equations

The respondent had failed to deal with \( \pm 5 \) and there is consideration of +5 with disregard that another value of \( x + 1 \) is -5. There is lack of procedural and conceptual knowledge on how to solve such a linear equation. There is a lack of meaning with the \( \pm \).

**Over-reliance on an algorithm**

All participants tried to use some algorithm and none tried inspection. Inspection or trial and error could have been used successful if the participants could deduce the processes that result in the equation and will be able to reverse it. One will need to find a number that multiplies by itself to get 25 and the possible values are 5 and -5 therefore \( x + 1 = 5 \) or \( x + 1 = -5 \). This is evident of failure to understand the structure of the equation.
Failure to realise that $\sqrt{25} = \pm 5$

Figure 36: 3A13 response to question 5 showing failure to consider -5 for $\sqrt{25}$

It was a challenge for some learners to realise that $\sqrt{25} = \pm 5$. There was failure to realize that the square root has two possible values. 3A50 was surprised that the other square root of 25 is -5 in the interview and requested some time to verify this. Reason for not considering the negative root is that they are not emphasized on the introduction to square roots at form 2. It is only the positive square roots that are emphasized.

In solving linear equation learners master the rule that what you do one side is what you do on the other so they fail to introduce the $\pm$ on the other side after taking the square root. Mastering the fact that what you do on one side is what you do on the other side has acted as an obstacle.
4.2.6 Question 6

Participants were asked to solve the problem: A square and a rectangle are equal in area. The length of the rectangle is 3 cm more than twice the length of the side of the square and the width of the rectangle is 2 cm less than the side of the square. Find the length of the side of the square.

Table 4.7 Categories of responses for question 6

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solved correctly with equation formulated correctly</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Formulated equation ((2x+3)(x-2)=x^2) but failed to solve correctly</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>Has 2 solutions (x = -2) and (x = 3)</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Some numerical values manipulated but no equation formulated</td>
<td>5</td>
<td>34</td>
</tr>
<tr>
<td>Wrong equation formulated</td>
<td>51</td>
<td>97</td>
</tr>
<tr>
<td>((2x+3)(x-2)) equated to zero</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Correct answer but wrong working</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>No solution</td>
<td>39</td>
<td>73</td>
</tr>
</tbody>
</table>

Of interest it was to note that learners at HW and KD all failed to solve the question. Word problems are a challenge as indicated by the teacher at HW. Literature confirms that word problems pose challenges for learners. There were a significant large number of no attempts that is 28% did not attempt the question.

Failure to make a connection

Equation was not formulated correctly as learners failed to make a link or connection with the given information. Even those with the expressions \(3 + 2x\) and \(x - 2\) could not conjecture the link between the given information. Learners failed to make the connection between the statements and area. In some cases \((2x+3)(x-2)\) equated to zero with a fatal implication that the area of the rectangle is zero.

Didis and Erbas (2015) indicated that students’ difficulties in solving word problems stem from difficulties in symbolizing meaningful relationships within algebraic equations. The text comprehension factor is the main issue for students in solving word problems as also reported by Cummins et al (1988) and Nathan et al (1992). Students make errors because they fail to see how
the situational aspect of the problem is related to formal expressions in their attempts to produce the intended internal representations. They also claimed that the major reason for students’ difficulties with word problems arises from not understanding the algebraic logic of a problem. Prior experience with arithmetic word problems contribute to difficulties as they will perceive the problem-solving process as a series of calculations and shift their thought processes from algebraic thinking to arithmetic thinking when solving algebraic word problems.

Figure 37: 3A13 response to question 5 showing failure to make a connection
HW44 and HW47 manipulated some values and the connection with the problem is difficult to follow.

**Keeping the two roots from the solution of the equation as the length of the square, a positive root and the other negative.**

A learner being interviewed said the two values of \( x \) must be kept the two since it is a quadratic equation and such equations usually have 2 roots so -2 and 3 were given as the length of the square. He could not relate the answer to the context. From the interviews it was clear that learners are not familiar with situations where quadratic equations are applied in real life so the questions about application were abandoned in oral interviews. It is a hurdle for low achieving students to solve word problems and solving problems on application is a challenge.

High achieving students from 3C and 3D performed better in the question. 3D27 used trial and error to solve \( 6 = x^2 - x \) and this shows the learner had process view of an equation. Such a view is also relevant to solving question 5.
Figure 39: 3D27 response to question 6 showing use of inspection to solve the problem

\[ x^2 = (2x + 3)(x - 2) \]
\[ x^2 = 2x^2 - 4x + 3x - 6 \]
\[ x^2 = 2x^2 - x - 6 \]
\[ x + 6 = 2x^2 - x^2 \]
\[ x + 6 = x^2 \]
\[ 6 = x^2 - x \]
\[ 6 = 3^2 - 3 \]
\[ 6 = 9 - 3 \]
\[ 6 = 6 \]
\[ x = 3 \]
3D42 has a zero product property misconception as his question 4.

Figure 40: 3D42 response to question 6 showing zero product property over-generalisation

\[(3 + 2x)(x - 2) = x^2\]
\[3(x - 2) + 2x(x - 2) = x^2\]
\[3x - 6 + 2x^2 - 4x = x^2\]
\[b = 2x^2 - 2x^2 + x\]
\[b = -x^2 + x\]
\[b = x + x - x^3\]
\[b = x(1 - x)\]
\[x = 6 \text{ or } 1 - x = 6\]
4.2.7 Question 7 (a)

Participants were asked to respond to the task:

(a) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of \( y = x^2 - 4x + 3 \).

(b) Hence write down the solution of the equation \( x^2 - 4x + 3 = 0 \)

The categories of responses are tabulated below.

Table 4.8 *Categories of responses for question 7(a)*

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>HW</td>
</tr>
<tr>
<td>Complete correct graph with table of values shown</td>
<td>65</td>
<td>0</td>
</tr>
<tr>
<td>Graph but no table of values</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Wrong form of graph due to points being plotted from a wrong table of values</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Joining points by a straight line</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Table of values only shown</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Graph limited to some range of values therefore not being u-shaped</td>
<td>21</td>
<td>7</td>
</tr>
<tr>
<td>Straight line</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>No solution</td>
<td>20</td>
<td>22</td>
</tr>
</tbody>
</table>

Errors and misconceptions observed are:

**Joining points by straight lines**

Literature also confirms that some over-generalise the concept of linearity to quadratic functions as seen in studies conducted by Zaslavsky(1997). Since learners meet linear functions before quadratic functions hence the tendency to make the over-generalisation on graphs of quadratic functions and join points by straight lines. The over-generalisation is explained by the constructivist theory which posits that learners create new knowledge by making a connection of the new material with prior learning experiences.
Wrong table of values resulting in wrong form of the graph

The graph should be a cup shaped parabola but some just followed their points resulting in the wrong form of the graph due to incorrect table of values. Learners should check their table of values so that they draw the correct form of the graph. If learners had the correct conception of the form of the graph of the quadratic function they would have checked their table of values after noticing the anomaly. The anomaly would create a cognitive conflict prompting a revisit of the table of values so that it conforms to the correct concept definition of the shape of the graph when the coefficient of $x^2$ is positive.

Some had a cap-shaped graph showing a limitation to the concept of the graph of a quadratic function.

Limiting the graph only to the positive values

Due to the misconception of having a pictorial view of the graph some learners only limited the graph to positive values only on the independent axis with no attempt to draw a complete parabola that reflects all characteristics of the graph of a quadratic function. The results agree with those from Leinhardt et al (1990), Ibeawuchi and Ngoepe (2012), Dreyfus and Einsenber (1983) and Zaslavsky (1997) where learners fail to appreciate that quadratic functions have a infinite domain.
4.2.8 Question 7(b)

The categories of responses are tabulated below.

Table 4.9 Categories of responses for question 7(b)

<table>
<thead>
<tr>
<th>CATEGORY OF RESPONSE</th>
<th>SCHOOL</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct answer written down and deduced from graph or table of values</td>
<td>33</td>
<td>33</td>
</tr>
<tr>
<td>Correct answer with an algebraic solution, though graph and table of values also displayed</td>
<td>30</td>
<td>37</td>
</tr>
<tr>
<td>Wrong or no answer from incorrect working but table of values and graph correct</td>
<td>26</td>
<td>41</td>
</tr>
<tr>
<td>Wrong answer from algebraic solution or no answer, table of values and graph also wrong</td>
<td>9</td>
<td>22</td>
</tr>
<tr>
<td>Wrong or no answer but correct table of values which can be used to deduce the solution, however a wrong graph drawn</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Correct answer, no graph or table of values</td>
<td>6</td>
<td>13</td>
</tr>
<tr>
<td>Wrong or no solution from algebraic method and no graph</td>
<td>5</td>
<td>23</td>
</tr>
<tr>
<td>Correct answer, wrong table of values, wrong graph</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>No solution</td>
<td>16</td>
<td>52</td>
</tr>
</tbody>
</table>

Errors and misconceptions observed in approaching this question are:

**Failure to deduce the connection between the table of values, the graph and the equation.**

Due to failure to make the connection between the table of values, the graph and the equation some learners went on to solve the problem algebraically which was unnecessary. 16 % even had no answer or wrong answer even though the table of values and the graph could be used to solve the equation and were correct.
The respondent failed to make the connection between the table of values, the graph and the equation, hence ended up with wrong answer. 3D21 said table of values could not be used solely as he responded that, “in mathematics one has to show working and you will not get all the
marks”, but had already obtained the answer with the graph and was able to explain how the graph could be used to solve the equation. 3D29 said working was to check the answer.

Learners are more comfortable with algebraic methods and this agrees with similar results from Knuth (2000). Only 13 % just wrote down the answer. Some learners did not attempt to make the connection between the 3 representations.

Some learners said that the graph could not be used as the correctness of the answer depends on the accuracy of the graph and this tendency not to rely on a graph could have resulted from earlier encounters where the graph does not cut the x-axis does on a definite point and such solutions could only be approximates.

**Use of algorithms or procedure without understanding**

Figure 42: 3A25 response to question 7b showing use of a procedure without understanding

![3A25 RESPONSE TO 7b](image)

Figure 43: 3A44 response to question 7b showing use of a procedure without understanding

![3A44 RESPONSE TO QUESTION 7B](image)
The above learners in an interview demonstrated the subtraction

\[
y = x^2 - 4x + 3 \\
- x^2 - 4x + 3 \\
0 0 0
\]

This shows that \( y = 0 \) but could not use the graph to solve the equation demonstrating the use of a procedure without understanding. He said that, “we were told by Mr [Name supplied] to subtract”. Learners lacked conceptual knowledge needed to understand why they were doing so.

Subtraction is done when solving the equation \( x^2 - 4x + d = 0 \) where the expression \( x^2 - 4x + d \) with \( d \) as a constant is such that \( d \neq 3 \).

From MC high-achieving learners in 3C and 3D were least likely to write a wrong solution if they have a correct table of values and graph. 17 % from 3C and 3D wrote a wrong answer compared to 33% from 3A and 3B, the low performing students.

**4.3 Results from the interviews**

Interesting results from the interviews are the following

**Students understanding of the symmetry of the graph of a quadratic function**

Students demonstrated good proficiency in understanding the symmetry of the graph of a quadratic function and when given half of the segment there were able to complete the graph applying symmetry well.

**Process view of the root of a quadratic equation**

Most students interviewed had a process view of the root of an equation with 3D43 giving a typical answer that the root is what we get after solving an equation. A quadratic equation was defined by one participant as an equation that gives 2 solutions when solved demonstrating that an action need to be performed on the equation to realise that it is a quadratic equation. 3A3 has this to say about a quadratic equation, “quadratic equation is like a regular equation but more modified and difficult”. This explains the tendency of some participants to cling to the linear equation schema and use the techniques of solving linear equations when solving quadratic equations.
Failure to explain why a quadratic equation has 2 possible roots.

All 31 participants interviewed could not explain why a quadratic equation has 2 possible roots due to failure to link the graph of a quadratic equation with the number of possible roots. Other participants linked the number of roots to the $\pm$ in the quadratic formula. Another participant actually said a quadratic equation has 3 roots which are “factorization, the quadratic formula and finding the perfect square”. Learners should move towards the conceptualizing the quadratic function as being a many-to-one function.

The case of 3C52

Below is a transcript of the interview with a learner coded as 3C52. He failed to define the root of an equation and his response to the use of the short formula was that it is easier.

Transcript of interview with 3C52.

Interviewer: What is a quadratic equation?

3C52: The equation that is multiplied by itself or squared to get the given equation

Interviewer: What do you understand by the root of a quadratic equation?

3C52: The number you get after solving the equation by factorization or quadratic formula or the graph.

Interviewer: How do you check that the root you obtain after solving a quadratic equation is correct?

3C52: You can check by giving the square of the value or substituting.

Interviewer: Why many possible roots can a quadratic equation have?

3C52: It has 3 roots.

Interviewer: Give an explanation to you answer

3C52: It is because the quadratic equation has 3 letters to be simplified or which you use to substitute.
4.4 Discussion of the results

This subsection is concerned with the general discussion and conclusions drawn from the findings obtained from the test and interviews.

Pupils displayed numerous errors and misconceptions in solving quadratic equations. Some were lacking in the procedural understanding of factorization as they failed to reverse expansion. This is from failure to expand algebraic terms correctly.

The researcher noted errors and misconceptions in simplifying algebraic fractions. This was seen when a participant would simplify \( \frac{x^2 - 2x}{2} \) to \( x^2 - x \) where division by 2 was only done on \( 2x \). This is due to incomplete understanding of fractions and arithmetic misconceptions.

The researcher also noted that learners did not have an object view of the root of an equation and as such they could not identify the roots as values that give you zero after substituting into the function when making a table of values. They could not pick the “zeros” of an equation and identify roots as values they would also get after solving the equation \( x^2 - 4x + 3 = 0 \) and their attempts to find roots is unidirectional. This was seen from learners who failed to write the solution to the equation on 7(b) but had a correct table of values and graph. They just needed to identify the \( x \)’s that give them \( y \) as zero. A challenge noted was failure to make the connection between the table of values, graph of a quadratic function and the solution to an equation.

A misconception was in over-generalization of the zero-product property. Pupils apply it without a clear understanding of why it works and they tend to over-generalize it. This is due to learners applying the rules without understanding. If \( a \times b = 0 \) then either \( a = 0 \) or \( b = 0 \) and this does not work if the RHS is not equal to zero.

A challenge noted was the failure to solve resulting linear equations. Pupils must be grounded firmly in tackling linear equations so that they can solve the resulting linear equations after factorization when solving quadratic equations.

It was also noted that when learners reach an impasse they invent rules albeit wrong ones to deal with the problem as noted when treating \( \sqrt{-12} \) as \( \sqrt{12} \).
Pupils need conceptual knowledge to be able to solve problems and their lack of was a hindrance in most situations. Procedures were used without understanding as in the attempt to solve question 7b where the learners would just recall that they were told to follow a certain procedure like subtracting \( y = (x^2 - 4x + 3) - (x^2 - 4x + 3) \) and obtaining zero but would not know how to proceed because of lack of conceptual knowledge of why they were doing so in the first place.

Recalling the quadratic formula by some learners is a challenge and is too long for them. Precision is required in mathematics as the short form lead to erroneous results.

There were some answers in the test which the researcher failed to pursue as no meaning could be drawn from them and in the interviews the participants would also explain giving odd answers and could not be discussed here.

4.5 Summary

This section summarise results from the chapter. Pupils had challenges and misconceptions in solving quadratic equations. Most errors were algebraic with pupils failing to simplify algebraic expressions correctly. The misconceptions could be traced to earlier learning experiences. The biggest challenge was in formulating an equation from a word problem as learners failed to connect the information.

It was also noted from the research the tendency to apply procedures and algorithms without understanding and for such learners their view of mathematics is instrumental without an understanding of why they are doing it. Most of the students in this study received their instruction from a curriculum that emphasizes procedural understanding (Mushava, 2016) and is the reason for the tendency to use algorithms and procedures in cases where there are unnecessary.

An observation that also has major implications and affects their learning of mathematics is the failure to recognize mistakes and make a correction. There was a general lack of recognizing a dead end by some participants. Application of Polya’s strategy to problem solving is suitable to recognize an inappropriate solution and learners need to examine the solution obtained and ask the questions, “Can you check the result? Can you check the argument? Can you derive the result differently? Can you see it at a glance? Can you use the result? Can you use the result or the method for some other problem? “(1957, pg. xvii)
CHAPTER 5
SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter will focus on the general summary, conclusion and recommendations of the results obtained from the research. Conclusions are guided by the research question: The nature of students’ errors and misconceptions in solving quadratic equations and their origin. The research project is summarized, conclusions guided by the research question are drawn from the findings and recommendations are made based on the study.

5.2 Summary

The research investigated the errors made and misconceptions learners have in solving quadratic equations. Possible sources of the misconceptions were explained. Form 3 pupils were involved in this research and they had done the topic at most 3 months from the time data collection was done. A case study approach was used. Sampling done was convenience sampling.

Data was collected through a test item and semi-structured interviews. 249 participants took the test item that was qualitatively analysed and 31 of the participants were selected for one to one interviews based on their responses from the test items.

A considerable number of errors, misconceptions and challenges were observed from the test and interviews. Most of the misconceptions were a result of over-generalisation, use of algorithms and procedures without understanding. If pupils cannot make connections with given information then challenges are likely to emanate and they will not be able to understand why and when a procedure is to be used.

It was observed that most pupils had a process view of the meaning of a quadratic equation and root of an equation. Some of the learners were operating at the pre-action level and were not ready to understand and use procedures and algorithms to solve quadratic equations.

Constructivism and the theories of the concept definition and concept image were used to explain the sources of misconceptions. The notion of relational and procedural understanding of mathematics was also used to explain the errors and misconceptions.

High achieving students were found from the interviews to most likely have an object view of the concept of a quadratic equation and a root of an equation whilst low achieving students were
found to have an action or process view of an equation. This study will improve instruction as teachers’ knowledge of the errors students are likely to make and misconceptions they are most likely to have when solving quadratic equations help to create instructional tasks that help students overcome those obstacles.

Over-generalisation was seen through failure to understand the zero-product property and use it correctly. A typical example is 3D39 who on solving $x^2 - 2x - 4=0$ had $x(x - 2) = 4$ giving the roots of the equation as $x = 4$ or $x = 4+2=6$. 52 participants had the misconception of over-generalising the zero-product property and it was exhibited in question 4. There was also failure to apply the zero-product property as seen in question 3 with 25 participants who on solving $7m^2 = 3m$ factorised correctly to $m (7m-3)$ but failed to deduce the correct solutions. The participants could not realize when to use the zero-product property. Misapplication of zero-product property was also seen in the solution of 3C19 in solving question 2. In solving the equation $4x^2 - 12x + 5= 0$ the learner had $4x^2 - 12x = -5$ with the left hand side factorised to $4x(x-3)$ and equated to -5. Now he had $4x = -5$ and $x-3 = -5$ leading to wrong solutions on solving the linear equations.

Misconceptions result from over-generalisations as seen in the solution of 3D1 who misapplied the distributive property on square roots. The learner had a belief that $(x+1)^2 = \sqrt{x^2} + \sqrt{1}$. There is also wrong expansion implied of $(x+1)^2$ to $x^2 +1$.

There were also attempts to linearise the quadratic equations and apply the techniques of solving linear equations. This was seen through attempts to isolate $x$ and make it linear and solve the resulting linear equations. The attempts were through taking square roots or dividing by the squared terms by the unknown so that they have linear terms. The linear equation schema was erroneously used for solving quadratic equations. 3B16 attempted to take square roots both sides of the equation $7m^2 = 3m$ so as to isolate $m$ and solve the resulting linear equations.

Over-generalising can also be attributed to the use of procedural knowledge without understanding. If students do not know why and when to apply a procedure then challenges are likely to emanate when faced with unfamiliar situations.
Some of the errors and misconceptions that permeated through the students in attempting the test items were algebraic malrules. There was failure by some students to distinguish like and unlike terms and therefore they could not identify which expressions can be simplified further. Misconceptions were also identified in simplifying algebraic fractions with a typical example by 3A8 where $\frac{x^2 - ax}{x^2}$ was simplified to $-ax$ and with 3A13 where $\frac{x^2 + 21}{x}$ was simplified to $x + 21$. The rule of division was misapplied and such simplification can only occur to a product, for example $\frac{y^2 (-ay)}{y^2}$ which is simplified to $-ay$. The learners had over-generalised the rule for the division.

The origin of the wrong quadratic formula was explored. The students realise that they have to divide by $2a$ at the end but could not recall that all terms are to be divided by $2a$ for example work by 3B34 and 3A37 who had $-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ as the quadratic formula. In some cases there was inconsistent division by $2a$ like KD15 who then divided all terms in the numerator at the end by the numerical value of $2a$ but started with the wrong quadratic formula. In the interviews one of the participants could not point out the differences in the correct and the wrong formula. In question 2, 45 of the participants had the wrong quadratic formula with 32 having the one that does not divide all terms in the numerator by $2a$.

An error that was also observed was imposition of $\sqrt{-x}$ as $\sqrt{x}$ with $\sqrt{1 - 25}$ given as $\sqrt{24}$ by 3B34 in questions 1 and 3. In question 34, 13 participants imposed $\sqrt{-x}$ as $\sqrt{x}$ with some cases of $\sqrt{-12}$ given the same numerical value as $\sqrt{12}$.

Other challenges noted were failure to factorise expressions correctly and solve the resulting linear equations demonstrating that some of the participants did not have the requisite knowledge to solve quadratic equations.

In the word problem on question 6 there were challenges in formulating the correct equation with 97 participants formulating the equation incorrectly. There was failure to make a connection with the given information and use the equal sign correctly. In some cases $(2x + 3)(x - 2)$, the area of the rectangle was equated to zero. 28% of the participants did not attempt the question. There
were some participants who manipulated numerical values with no equation formed. The calculations were meaningless leading to a wrong solution. There is a confirmed tendency for learners to concentrate on the numerical calculations or manipulations disregarding how the connections are intertwined, Ghazali(2011) cited in Parent(2015). There were some participants who kept the two roots of the equation \((2x + 3)(x - 2) = x^2\) which are -2 and 3 as the possible values of \(x\) disregard their dealing with the length of the side of a square which cannot be negative. Their justification for keeping the two values was that it is a quadratic equation so it must have two roots.

In drawing the graph of a quadratic function some participants joined points by straight lines showing the use of the linear function schema which had to be modified in this case. The learners had not accommodated the quadratic function schema. Some participants limited their range with the result that the graph was not a parabola indicating a limitation on the knowledge of quadratic functions.

Failure to make connections or deduce information from table of values and the graph was shown in question 7(b) where the participants were to use a graph to solve an equation. 30.9% of the participants could not use the graph or the table of values to find the solution to the equation. They went on to do some algebraic calculations though the table of values and graph could be used to give the solution and the question explicitly asked them to deduce the solution from earlier work. There is an inclination to use algebraic methods disregarding the graph and this agrees with similar findings from Knuth (2000). One of the respondents (3D21) in the interview said he did the calculation because he believed that in mathematics one has to show the method, though he knew how to use the graph to solve the equation. 41 of the 259 participants did an incorrect working leading to a wrong answer though the graph and the table of values were correct showing that some of the participants did not know how to use the graph to solve an equation.

Procedures were used without understanding as seen with the overgeneralization of the zero product rule and use of an algorithm in answering question 7(b) demonstrated by 3A25 and 3A44 where they made an unnecessary subtraction in order to find the roots of the equation. They claim that they have been taught to do so but in this case it was unnecessary because for the
equation \( x^2 - 4x + 3 = 0 \) the graph was that of \( y = x^2 - 4x + 3 \). It would have been necessary if the graph was that of \( y = x^2 - 4x + d \) where \( d \neq 3 \).

Sources of errors and misconceptions can be identified as overgeneralization of earlier learned procedures, use of procedures without understanding that is lack of conceptual knowledge, failure to recall the quadratic formula, failure to accommodate the quadratic equation schema with learners holding on to the linear equation schema and a poor background in fractions, simplifying algebraic expressions and factorisation.

5.3 Conclusions

My goal in conducting this research was to gain insight into the errors students make and the misconceptions they have when they solve quadratic equations. Theories of constructivism, procedural and conceptual knowledge, concept definition and concept image informed the research project. Errors and misconceptions were identified from the 249 participants who were enrolled at 3 schools in Mashonaland West Province of Zimbabwe. Recommendations suggested by the study are then cited for future teaching, curriculum planning and research.

Notable errors and misconceptions are the overgeneralization of the zero product property, dealing with algebraic expressions and fractions, tendency to linearise quadratic expressions and use techniques they have learnt about linear equations to solve quadratic equations and joining points of the quadratic graph by straight lines.

A surprising encounter in the participants work was students giving the value of \( \sqrt{12} \) for \( \sqrt{-12} \) and it was not clear on what logical basis they have done so or the interconnection made. This shows a general lack of conceptual knowledge in the students’ work. Errors were due to such malrules invented by the students when trying to deal with an unfamiliar situation.

There is also overtendency to use the quadratic formula to solve equations at HW, one of the schools where the research was done and in cases where it is unnecessary as a shorter and easy method would have been appropriate.

Challenges noted were in factorizing quadratic expressions and solving linear equations. In factorizing quadratic expressions some of the participants just concentrated on the term with \( x^2 \).
which was the leading term. For such learners they are at the pre-action level for solving quadratic equations as they lack the necessary background knowledge. Another challenge is failure to recall the quadratic formula.

Most of the participants could not make use of the information about root of an equation to solve a problem as given in question 1. The problem was acknowledged to be unfamiliar by the teachers at the 3 schools during the test validation phase. It was not a standard question which departed from the familiar textbook problems and this created challenges as also pointed out by Schoenfeld(1985) with such unfamiliar questions.

On solving an equation by a graph some of the participants could not deduce the information that a graph and a table of values gives. In teaching learners can be guided to describe the properties of the graph from a table of values.

It is important for learners to have conceptual knowledge about a task for a topic and this helps them to know why and when a procedure can be used. This also allows them to use multiple strategies. I will also encompass this in my teaching so that there is no overreliance on drill and practice techniques.

5.4 Recommendations

From the findings in the study the researcher recommends the following in teaching learners to solve quadratic equations

- In the teaching of quadratic equations teachers should put emphasis on the meaning of the root or solution of an equation and assist learners to have an object view of the root of an equation. Learners failing to find roots may be that in their learning experience some of the concepts were not addressed and teachers should create situations where learners explore the meaning of a root of an equation.

- Learners should be encouraged to find the roots from their understanding of the equations so that they can deduce them from inspection, for example with the equation \((x+1)^2 = 25\), learners may start by deducing that the number which multiplies itself to get 25 is 5 or -5 and so \(x + 1 = 5\) or \(x + 1 = -5\) and solve the resulting linear equations. Learners need an object view of an equation to be able to do so and de-encapsulate the object into a form it
was prior to the transformation and this was demonstrated by 3D27 who used trial and error to solve $6 = x^2 - x$.

- Multiple strategies to solve quadratic equations should be taught so that learners do not overly rely on one method which can be a burden in some situations. The reliance on ritualistic procedures was also noted by Knuth (2000) as students would engage in extra procedural steps but a more direct route would have been appropriate. This is due to the fact that students are more likely to use mathematical procedures rather than know how the mathematical procedures are achieved Ghazali (2011) as cited in Parent (2015). Students focus much on calculation procedures rather than finding out how the conceptual pieces are connected.

- In teaching graphs connections should be made between the table of values, the graph and the equation.

- Learners should be encouraged to check their answers and relate them to the given problem. Learners should be able to realise the futility of their attempts and recognize dead ends. If there is a dead end there is need to restrategise the solution or check the calculations as some ended up with the square roots of negative numbers which they have not encountered and did not know how to deal with. A retracing of the steps and checking numerical computations and substitution would have solved the problem in some of the cases.

- Problem solving strategies suggested by Polya (1957) should be encouraged so that learners validate the appropriateness of their solution.

- Teachers need to find ways to identify and deal with pupils’ misconceptions in other areas like arithmetic and fractions as to assist learners understand higher level mathematics as learning mathematic is hierarchical.

- Pointing out shortfalls in students’ expressions such as in the quadratic formula during teaching will help them remember it correctly and foster precision in expressions. I would suggest a curriculum that emphasise that learners derive the quadratic formula as to derive meaning in the expression and not just memorise it. Instructional tasks should be ones that create conceptual understanding of the quadratic formula.
5.5 Recommendations for further areas of study

- Pedagogical content knowledge held by teachers in teaching quadratic functions and equations.
- Finding the connection between instruction that emphasizes conceptual understanding in quadratic functions and equations and students’ performance.
- Research on the misconceptions about topics that link with quadratic equations, that is misconceptions in algebraic expressions, formulation of algebraic expressions, misconceptions about graphs or how students view graphs.
- Another potential area of research is exploring the cause of the disconnection between students’ ability to graph a quadratic function and using it to find the solution to equations. Research will be on students’ perception of the graph of a quadratic function and how it relates to their ability to use the graph to solve a quadratic equation.
REFERENCES


Confrey,J. (1990). A review of the research on students’ conception in mathematics, science and


APPENDIX A: APPROVAL LETTER FROM THE MINISTRY OF EDUCATION

All communications should be addressed to "The Provincial Education Director"
Telephone: 067-23043
Tele Fax: 067-23320
Email edumashwest@gmail.com

Ref: C/246/1/MW
Ministry of Primary & Secondary Education
P.O Box 328
Chinhoyi

Mr/Ms Mashudzi L
Kutama College

Dear Sir/Madam

APPLICATION FOR PERMISSION TO CARRY OUT AN EDUCATIONAL RESEARCH:
SCHOOLS IN MASHONALAND WEST PROVINCE: AREA OF RESEARCH:

Your application letter dated 10/16 seeking authority to carry out a research/survey in schools in Mashonaland West Province refers:

Permission has been granted by the Provincial Education Director on the following conditions:-
- that the learning and teaching programmes at the targeted schools are not interrupted in any way.
- that you strictly adhere to the activities and topics specified in your letter of request.
- that the permission or authority may be withdrawn at any time by this office or a higher office if need be.

Please apprise this office on your research findings for the benefit of the Province.

By this letter, all District Education Officers and Heads of schools you wish to visit are kindly requested to give you assistance in your work.

We wish you success in your research and studies.

For: PROVINCIAL EDUCATION DIRECTOR
MASHONALAND WEST PROVINCE

19 OCT 2016
HUMAN RESOURCES
MASH. WEST PROVINCE
P.O. BOX 328, CHINHOYI
ZIMBABWE
APPENDIX B: TEST

INTRODUCTION

This questionnaire is part of the items to be used in educational research on students understanding of quadratic equations. It is not part of the common assessment criterion.

INSTRUCTIONS

1. Answer all questions.

2. Use the answer space for working and show all working where appropriate.

3. Answers that need rounding off give them to 3 significant figures.

4. Where necessary use the graph paper provided.

5. Do not write your name

QUESTION 1

If one of the roots of the equation \( x^2 - ax + 21 = 0 \) is 3, what is the value of \( a \)?

SOLUTION
QUESTION 2
Solve the equation $4x^2 - 12x + 5 = 0$

SOLUTION

QUESTION 3
Solve the equation $7m^2 = 3m$

SOLUTION

QUESTION 4
Solve the equation $x^2 - 2x = 4$

SOLUTION
QUESTION 5
Solve the equation \((x + 1)^2 = 25\)

SOLUTION

QUESTION 6
A square and a rectangle are equal in area. The length of the rectangle is 3 cm more than twice the length of the side of the square and the width of the rectangle is 2 cm less than the side of the square. Find the length of the side of the square.

SOLUTION
**QUESTION 7**

(a) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of \( y = x^2 - 4x + 3 \).

(b) Hence write down the solution of the equation \( x^2 - 4x + 3 = 0 \)

**SOLUTION**

(a) On graph paper

(b)
APPENDIX C: INTERVIEW GUIDE

INTRODUCTION: Thank you for taking your time to answer questions about quadratic equations and functions. I am researching on how students solve quadratic equations and think about quadratic functions.

Question 1: What is a quadratic equation?

Question 2: What do you understand by the root of a quadratic equation?

Question 3: How do you check that the root you obtain after solving a quadratic equation is correct?

Question 4: Why many possible roots can a quadratic equation have?

Question 5: Why is it possible for a quadratic equation to have 2 roots?

Question 6: What is the name given to graphs of a quadratic function?

Question 7: If the graph of a quadratic function is cut into half sketch the other half so as to complete the graph.
Question 8: How is the graph of $y = x^2 - 4x + 3$ used to solve the equation $x^2 - 4x + 3 = 0$?

If student used any algebraic approach and got question 6 correct ask why they decided to use the algebraic approach instead of just using the graph. If there is no graph of the function $y = x^2 - 4x + 3$ ask how it can be drawn.

Question 9: How is the graph of $y = x^2 - 4x + 3$ used to solve the equation $x^2 - 4x + 3 = 2$?

If in the test item a wrong quadratic formula was used ask the participant to write down the quadratic formula and ask for variations to the standard formula and how the respondent considers them.

Thank the student for taking part in the interview.