Secondary School Students’ Errors and Misconceptions in Learning Algebra

BY

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30 SEPTEMBER 2016
DECLARATION:

I, Osten Ndemo, declare that this project is my original work and affirm that it has not been submitted to this or any other university in support of an application for a degree or any other qualification.

Signed______________________

Date________________________

Supervisors: I ___________________________ and ____________________________

declare that we have supervised this thesis and we are satisfied that it can be submitted to the faculty of Science Education, Bindura University of Science Education.

Signed: __________________________

Date: ____________________________
DEDICATIONS:

To my dear wife, Mary Yeuka,

For her love, that is transcending time and space and encouragement,

To my children, Blessing, Blessmore, Believe and Benevolence,

For their patience, support and understanding.
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The study explored secondary school students’ conceptions of algebra, the kinds and sources of errors and misconceptions in this domain. The desire to unearth why students fail mathematics, with particular focus on school algebra, motivated this study. Systematic random sampling was used to draw sixty-five participants from a population of two hundred and twenty-three form three students. A case study was employed and data was collected through administering written tests, a structured questionnaire and interviewing the students. Content analysis of students’ written tests, questionnaire responses and interview transcriptions constituted data analysis. Types of errors and misconceptions were tabulated under the five conceptual areas that included variable and expression. Inferences about students’ conceptions of algebra were supported by students’ narrations and scanned solutions. The main findings were errors are common in students’ written work, errors and misconceptions are a result of mathematical thinking on the part of students and students have limited understanding of the nature of algebra; in particular the notion of a variable is not well grasped. Regarding errors and misconceptions, conceptual errors and procedural errors were prevalent. The abstract nature of algebra was identified as the main source of errors. Errors are rooted in students’ minds and are difficult to dislodge. Consequently, mathematics educators should embrace errors and engage with them. Teachers should shift their minds from understanding of students’ errors as obstacles to learning mathematics to understanding errors and misconceptions as integral to learning and teaching mathematics. Further research can be carried on systematic errors.
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Chapter One

1.0 Introduction
An introductory road map through this text can be useful, especially for the readers to make the trip through the field of mathematics education research. This chapter gives a background to the study and a statement of the problem. It also covers three research questions and assumptions. The chapter entails the aims and purpose of the study. The significance, limitations and delimitations of the study are also covered in this chapter. Lastly, operational definitions of key terms and a summary of the whole chapter are found in this chapter.

1.1 Background to the study
There is widespread interest among nations in improving the levels of mathematics achievement in schools. Mathematics continues to command an enviable position in our everyday life with Tecla (2007) emphasising that in our march towards scientific and technological advancement we need nothing short of good performance at all levels of schooling. Mandevvu (1996) stated that most employers in Zimbabwe expect a job-seeking school leaver to have passed mathematics among the five subjects at ordinary level. It therefore means that the position of Mathematics in secondary school curriculum in Zimbabwe is important for scientific and technological development.

The poor performance in mathematics is an issue that has been tackled by many people in Zimbabwe (Mtewa, 2011). Skemp, (2008), reveals that the failure rate in mathematics in Zimbabwean schools is unacceptably high. Bush (2009) also cites several studies pointing to high failure rates in mathematics. Mass failures in mathematics need critical attention to achieve the aims stipulated in the Zimbabwe General Certificate of Education (ZGCE) for examination in November 2012 – 2017 “O” level mathematics syllabus (4008/4028).

The National Council of Teachers of Mathematics (NCTM, 2000) identifies a comprehensive foundation of mathematics competencies and understandings for all students. These standards are descriptions of what students should know and be able to do. Algebra is a branch of mathematics that was developed as one of the content standards students should know. The standards apply to all levels. However, emphasis on these standards varies from level to level.
For example in primary grades, greatest emphasis is on number sense while at secondary school level the greatest emphasis is on geometry and algebra.

The importance of learning algebra is widely acknowledged. Gillian (1990) affirms that abstract algebra is important in the education of mathematically trained person. The algebraic related terms like variables, expressions, functions, inequalities and equations are used ever widely in scientific disciplines like engineering and algebra still commands a central role in advanced mathematics. Algebra concepts are widely used in mathematics and other disciplines like engineering. A superficial knowledge of algebra will affect understanding of vast number of mathematics and scientific disciplines.

Thompson (2005) notes that mathematics curriculum is a collection of activities from which students may construct mathematical knowledge and that it is a sequence of activities, situational contexts, from which students construct a particular way of thinking. The study of algebra has led to a greater understanding of other branches of mathematics. The dependence of other mathematical disciplines on algebra shows that students can construct knowledge from other mathematical disciplines using it.

Although educators are aware of the importance of learning algebra, many of them report difficulties on the part of students in understanding algebraic ideas. Algebra provides a powerful means of studying various mathematical structures in abstract form before applying the results to particular situations when they arise. The branch of mathematics called algebra is very rich in symbols. The language of mathematics consists of symbols, terminology, notations, conversions, models and expressions that can only be interpreted by a mathematically literate person. Students find it difficult to comprehend key concepts like variable, expression, function, inequality and equation. Thus, educators try to find ways to help students understand abstract concepts in algebra and look for relevant ways to introduce these concepts to students. Gillian (1990) and Heistein (1986) believe that students will best appreciate the abstract theory when they have a firm grasp of what is being abstracted.

1.2 Statement of the problem

Despite the significance of algebra in school mathematics curricula, many students still find it difficult to comprehend (Witzel, Mercer & Miller, 2003).
Many attempts to better prepare students for algebra has not yielded greater achievement.

Greens and Rubenstein (2008) claim that the study of algebra until recently was reserved for college-bound students. Furthermore, students still struggle with algebraic concepts and skills, (Greens and Rubenstein, 2008) and many are discontinuing their study of advanced mathematics because of their lack of success in algebra. Secondary school students often make errors and experience misconceptions in learning algebra. The students’ lack of deep understanding of school algebra spurred this study to explore the nature of algebra and identify kinds and origins of errors and misconceptions in this domain. There is need for critical attention to unearth students’ conception of algebra and make an in-depth analysis of kinds and sources these errors and misconceptions in learning secondary school algebra to promote deep learning and engagement. It is a challenge to help students overcome their frustrations, but at the same time, it is necessary effort because of the important role mathematics plays in the world (Gatawa, 2008).

1.3 Research questions

The study will shed light on the following over-arching questions:

1 How do secondary school students conceive algebra?

2 What kinds of errors and misconceptions do students experience in learning school algebra?

3 What are the sources of errors and misconceptions in the domain of school algebra?

1.4 Assumptions

The researcher believed that the research participants completed the questionnaires honestly and truthfully, without assistance or delegating other people and they were in a stable mental and emotional state. In addition, the researcher assumed the tasks given to the participants were completely novel or new to them. When an error or similar errors occurred more than once in different situations, then the student might have a misconception. The students presumably had covered topics in algebra and they had gained literacy and numeracy skills. In addition, students were not affected by their closeness to the teacher, for instance the need to impress the teacher.
1.5 Aims and objectives of the study

The study dealt with an area of learning difficulties in mathematics. The study aimed to generate insights on students’ understanding of the nature of algebra. This study aimed to identify and categorise errors and misconceptions made by students in their attempts to solve algebraic problems. It would also shed light on possible sources of errors and misconceptions in the domain of algebra. The study’s fundamental objective was to provide pedagogical intervention and support with examples of students’ work in an attempt to provide teachers with pedagogical content knowledge necessary for implementations. Mathematical errors provide valuable insights for teachers into the students thinking as well as for students themselves (Hebert & Carpenter, 1992, p. 89). In the affective domain, the researcher hoped to kindle in students a deep love for mathematics and a feeling for the way mathematics should be determined.

1.6 Purpose of the study

One of main concerns of mathematics teaching is the improvement of students’ learning and understanding of mathematics. The purpose of the study was to develop a theory to explain why so many students fail to learn mathematics. The study focused on exploring why students make errors and experience misconception in learning school algebra.

1.7 Significance of the study

There has been considerable research in secondary school students’ difficulties in learning algebra. This dissertation provided an overview of the current literature pertaining to error analysis and misconceptions in algebra and possibly provided potential solutions to remedy algebraic misconceptions. This study may add more findings to the results of scholarly work by Kieran (1989 and 1992) and Matz (1982) who identified and classified errors by introducing relative frequency of errors made.

In the past, research on student errors and misconceptions was limited to the study of isolated concepts, such as variables, expressions and or equations. However, comparatively there are few studies to understand the combined effects of misconceptions and their interrelatedness pertaining to a number of areas. Algebraic concepts like other areas of mathematics are closely related.
The students’ errors and misconceptions could be viewed better if the five concepts are studied together and examined the relationships among the error patterns in a single study.

Revealing student errors and misconceptions is crucial, as students not provided with opportunities to experience algebra may continue to struggle throughout school life (Smith, et al., 1993). Errors and misconceptions analysis can prove to be a useful tool in the process of learning, as according to Labinowich, (quoted in Brooks, 1993:83), a child’s errors are actually natural steps to understanding. Further analysis of these errors and misconceptions might facilitate the design and development of improved teaching-learning strategies relevant to the Zimbabwe classroom situation. Teachers have potential to target more students and increase those students’ conceptual understanding of algebra by learning how to correct misconceptions rather than their mistakes (Ball, Sleep, Boers & Bass, 2009).

Explorations of secondary schools students’ errors and misconceptions were a crucial attempt to improve the quality of mathematics education. Knowledge of students’ cognitive obstacles to learning might enable mathematics teachers to deliver high quality instructions to all students. The study might provide an overview of essential processes involved in learning concepts in secondary school algebra. It might provide a dynamic conceptualisation needed to bridge the knowledge gaps in basic secondary school algebra.

Furthermore, the research might lay a firm pre-calculus foundation for learners and improve their advanced mathematical thinking. Students who hold misconceptions in secondary school algebra may continue to struggle with algebra later (National Mathematics Panel, 2008). Furthermore, the research supports the idea that teachers may need professional development in the domain of algebra to improve students’ algebraic teaching skills. It may also illuminate ideas that can contribute to improving the teaching and learning of mathematics in general and secondary school algebra in particular. It may broaden mathematics teachers’ understanding of algebra since they are also learners.

Mathematics knowledge is interrelated and misconceptions in one branch of mathematics may affect other areas of mathematics. Mathematics knowledge is cumulative. It builds on knowledge from previous mathematics lessons, back dated as far as elementary mathematics. Generally, on transition from primary school to secondary school most students fail to regard their previous knowledge.
A far more valuable aspect of Piaget’s theory is the transition from one mental state to another. A shaky foundation and mastery of basic concepts in algebra may limit a student’s further pursuance of other mathematical and scientific disciplines. Above all, algebra is a core course in secondary and undergraduate mathematics curricula.

Ideas in this study may inform teachers, curriculum planners and policy makers to broaden their understanding of how to identify errors and misconceptions and thoughtfully engage in error reduction. The understanding of such errors and misconceptions may shed light on how deep learning of algebraic concepts may be impeded. Significantly, findings from this study might generate ideas that may be useful and usable by mathematics classroom practitioners as they engage in mathematics education. The study might inform teachers on how to remedy errors and misconceptions. Overall, secondary school students’ mathematical understanding can be enhanced when they have deep algebraic understanding and proficiency.

1.8 Limitations of the study

These are uncontrollable aspects of the study that weaken its design and the generalisability of its findings. The participants were from the same geographical location. A more diverse group that excludes geographical bias could arguably cast more light on the research.

The case study only represented portraits of selected students in Masvingo district. It could not claim to have captured the entire performance in algebra. In order to minimise this aspect, the researcher employed a student-questionnaire, student-interview and tests as a multi-dimensional approach to data collection to ensure richness of data gathered. Another limitation was that the research could not explore the nature and sources of systematic errors since the test was not repeated. Fortunately, this study explored other categories of errors in algebra.

1.9 Delimitations of the study

The researcher carried out a case study at one Secondary School in Masvingo province. The school is a mission boarding school for boys and girls. It has classes from form one to form six. Form one to form four have four classes each. The students were streamed according to ability using form two end of year results. The target population for the study were form three students for 2015.
Mathematics is compulsory subject for all students at this school. As a school policy, every candidate has to sit for ZIMSEC ordinary level mathematics examination.

The study focussed on errors committed and misconceptions held by form three students in learning algebra. In exploring the students errors and misconceptions, the study was illuminated by focussing on five pillars or basic building blocks of algebra, namely variable, algebraic expression, function, equation and inequality at ordinary level. The case study might create a baseline for secondary schools in Masvingo province.

1.10 Definitions of key terms: Secondary school, Algebra, Errors, Misconceptions.

Secondary school: In this study, a secondary school is any post primary school other than a correspondence college, recognised by the Ministry of Education, offering post primary education to adolescent boys and/or girls from form 1 to 4.

Algebra: Algebra is a branch of mathematics that uses letters to represent numbers and uses the four operations in arithmetic; addition, subtraction, multiplication and division. It meant the study of a variable, symbolic expressions, equations, functions and inequalities.

Error: An error is a challenge that a student displays in response to mathematical tasks. It can be a skip, blunder, inaccuracy or a deviation from accuracy. For example, an error in simplifying $2 + x$ the student gives the answer as $2x$.

Misconceptions: A misconception is erroneous, illogical, or misinformed knowledge. It is inaccurate, incorrect or misunderstanding of an algebraic concept. For instance, some students have a misconception that Pythagoras theorem applies even when the triangle is non-right angled. They think the longest side in a triangle a hypotenuse: $x^2 = 3^2 + 2^2$

![Figure 1](image-url)
1.11 Summary

The chapter discussed the background to the study. It portrayed mathematics as a fundamental subject in our lives, for scientific and technological advancement of nations and algebra is one of the key content standards secondary students should know in mathematics. This chapter touched on the statement of the problem. Performance in mathematics has always been perennially dismal. Three research questions spurred this study. Assumptions, aims and objectives of the study featured in this section. The researcher explained the purpose and significance of the study. This section contains limitations and delimitations to the study. Operational definitions of key terms in the study; algebra, error, misconception and a secondary school were given. The next chapter discusses the literature review related to the current study, focussing mainly on the theoretical and conceptual frameworks of this study.
Chapter Two

Literature Review

2.0 Introduction

This chapter focuses on the theoretical framework guiding this study, with due considerations to exploration of students’ errors and misconceptions in learning school algebra. It covers the nature of algebra and its five major underpinnings. This chapter also discusses perspectives on the learning and understanding of algebra, kinds and sources of errors and misconceptions in the domain of school algebra. It carries a literature review of studies related to the current research. This section closes with a summary of the whole chapter.

2.1 Theoretical Framework

This study used constructivism as a main theoretical framework and as a lens to view and illuminate the nature of algebra and the kinds and sources of errors and misconceptions that students display when learning algebra. Constructivism is the most compatible, consistent and appropriate way to study human cognition. The language of constructivism is omnipresent in modern pedagogical theory and practice. Such a viewpoint in learning has been central to the empirical and theoretical work in education (Steffe & Gale, 1995). It has contributed to mathematics reform efforts (NCTM, 2007).

The learning of mathematics is a constructive process. Mathematical knowledge is constructed from related knowledge the student has. It is the role of the mathematics educators and policy makers to provide links between existing knowledge and the new knowledge of mathematics. Constructivism posits that learners actively construct and reconstruct their own experiences (Dengate & Lerman, 1995, p. 37; Prince & Felder, 2006, p.3-5). In the spiral of constructivism, concepts are shaped in the learning process. Constructivism derives from Jean Piaget’s assertion that conceptual knowledge cannot be transferred from one person to another (Piaget, 1970). Rather, each person based on personal experiences must construct it. The assumption is that students could explain their reasoning based on knowledge they constructed through their experiences in the classroom and learning process. Students come to the classroom with different experiences from their social lives.
Therefore, in the teaching-learning situation new knowledge can be constructed by sharing these experiences (von Glasserfeld, 1989b).

According to the constructivist perspective, effective learning occurs when students are actively involved in the learning process. Exposing students to solving real life mathematical tasks is essential in developing problem solving skills. One of the most important factors for improving performance is students’ involvement (Polya, 2011). By involvement, it means how much time, energy, effort students devote to the learning process (Mtetwa, 2011). There is a good deal of research evidence to suggest that the more time and effort students invest in the learning process, the more intensely they engage in their own education, the greater will be their growth, achievement, their satisfaction with their educational experiences and their persistence in school and more likely they are to continue their learning (Umameh, 2011). Students are unlikely to learn unless they are somehow involved in the process of learning; they seldom learn if they are treated as simply as passive receptors (Obodo, 2012).

The ZIMSEC “O” level syllabus emphasises a deliberate attempt to include problem-solving methodology. Polya (2011) posits that there are four phases through which a problem-solver proceeds in order to solve a confronting problem successfully. These are sequentially, understanding the problem, devising a plan or deciding on an approach for tackling a problem, executing a plan, looking back at the problem, the answer and how it was obtained.

From multiple research efforts on creating a constructivist classroom, Yackel et al. (1990) concluded that not only are students capable of developing their own methods for completing mathematics tasks but that each student has to construct his or her own mathematical knowledge. They develop mathematical concepts as they engage in mathematical activities including trying to make sense of methods and explanations they see and hear from others. However, the classroom too often separates the student’s conceptual knowledge from the new procedures or knowledge they construct because the students’ informal ways of making meaning are given little attention (Cobb, Yackel and Wood, 1991). Constructivists assert students actively construct their individual mathematical worlds by reorganising their experiences in an attempt to resolve their problems (Cobb, Yackel & Wood, 1991).
The expectation is that the students’ reorganised experiences form a personal mathematical structure that is more powerful, more complex and more abstract than it was prior to the reorganisation (Davis et al., 1995).

A fundamental principle underlying constructivist approach to learning mathematics is that a student’s activity and responses are always rational and meaningful to themselves, no matter how bizarre or weird they may seem to others. One of the teacher’s responsibilities is to determine or interpret the student’s rationality and meaning (Yackel et al., 1990). Before constructivism, teachers often had negative feelings about the mistakes the students would make regarding them as unfortunate events that need to be eliminated and possibly avoided at all times. Students make errors in the process of constructing their mathematical knowledge. However, regarding errors as valuable sources of students’ thinking replaced the strategy of more drill and practice. As teachers, it is difficult to escape from students’ mistakes, so it is worthwhile finding out why students make the mistakes in the first place and often continue to repeat the same mistakes. Mistakes or errors become entrenched, so error analysis is the first step towards doing something relevant to remove the cause of the errors.

Some schools of thought argue that if everyone had a different experiential world, no one could agree on any knowledge. The argument is that constructivism is a stance that denies reality (Kilpatrick, 1987). However, constructivists refute this criticism by saying that the agreement on social and scientific issues does not prove that what is experienced has objective reality. The models that we construct are our constructs that are accessible to us (von Glaserfeld, 1991). Similarly, if everyone is able to construct their own knowledge, then everyone’s constructs must be equally valid. The constructivists dismiss this criticism by saying that the constructive process does not happen in isolation, but is subject to social influences. The construction of knowledge is both social and individual (Kitchener, 1986).

2.2 Nature of algebra

There are many conceptions regarding the domain of algebra in the literature. Many algebraic concepts are found in the current secondary school curricula. The structural features of algebra in terms of variables, symbolic expressions, algebraic equations, functions and inequalities are connected together to form a broader conception of algebra.
It shows how algebra is related to other branches of mathematics like calculus and analytic geometry. These ideas are useful when selecting and including algebraic concepts in a test for students in secondary school.

Usiskin (1988) described the four possible meanings of a variable based on the views of algebra or fundamental conceptions of algebra. If algebra is as generalised arithmetic, then variables are thought of as pattern generalisers, for example the commutative characteristic of addition. If algebra is viewed as procedures for solving problems, a variable is viewed as an unknown, which is clearly related with equations. If algebra is viewed as the study of relationships, a variable is understood as argument, that is, it stands for a domain value of a function or a parameter or stands for a number on which other numbers depend. If algebra is viewed as the study of structures then the variable is understood as an arbitrary symbol.

The variable

A variable is hard to define because its definition largely depends on the context (Gunawaden 2011). Variables can be used in identities, formulas, equations, properties or parameters. A variable is a symbol, which may represent any one of the members of the specific field set called the replacement set or domain of the variable. The members of the domain are called the values of the variable. A variable with just one value is called a constant.

The first conception of algebra accepts that algebra is generalised arithmetic. Algebra is the study of structures, relations, equality and by substituting concrete numbers, it generalises arithmetic. Algebra deals with generalised numbers or variables and the relationships among these changing quantities (Booth, 1986). In this conception, a variable is a pattern generaliser. For example, the arithmetic expressions like \(-3 \times 2 = -6\) could be generalised to give a property like \(-x \times y = -xy\). The commutative property of addition \(3 + 2 = 2 + 3\) could be generalised to \(x + y = y + x\) in algebra. Thus, algebra is generalised arithmetic. Algebra has been transformed into many forms of mathematics like analytic geometry and calculus because of the power of algebra as generalised arithmetic.
The algebraic expression

A letter is used to build an expression. Either one letter or a combination of letters can form an algebraic expression.

When letters are used together with numbers and arithmetic operations, (+, −, ×, ÷) they form algebraic expressions, like 3\(y\), 2\(x\) + 3\(y\) (Kireri, 2008).

The second conception accepts algebra as the study of structures. In this notion, a variable is a little more than an arbitrary symbol. The variable is an arbitrary object in a structure related by certain properties. This is the view found in abstract algebra. For example, when factorising the problem 2\(x\) − 4\(y\) the conception of a variable is not an unknown. This brings the notion of algebraic expressions. The key instructions could be factorise or simplify.

The function

A function is any pairing of the members of one set (the domain) with the members of another set (the range) so that each member of the domain has exactly one partner in the range. Precisely, a function is a rule that uniquely associates each and every element of one set with a member or members of another set. This means every element of the domain is mapped to an element of the range and the image of any element in the domain is unique. In other words each and every element of the domain must be mapped to one and only one element of the range.

The third conception accepts algebra as the study of relationships. A variable is understood as argument, that is, it stands for a domain value of a function or a parameter or stands for a number on which other numbers depend. For example, a formula for the area of a rectangle is \(A = LW\). this is a relationship among three quantities. There is no feeling of an unknown here. Instead all \(A\), \(L\) and \(W\) can take many values. Similarly given the formula \(y = x + 2\) can be written in functional notation as \(f(x) = x + 2\). This is a relationship of two quantities. There is no unknown. In such cases, no solution process is involved.

The equation

Another important idea in algebra is equality. An equation has two or more expressions combined by an equal sign.
To solve an equation correctly, the student must know the rules or procedures of simplifying algebraic expressions.

In the fourth conception, algebra is viewed as procedures for solving problems and a variable is viewed as an unknown, which is clearly related with equations. This conception suggests that algebra is the study of procedures for solving certain kinds of problems. In this conception there is need to find a generalisation for a particular question and solve it for the unknown. Consider the problem, “When 3 is added to 5 times a certain number the result is 40. Find the number.” (Usiskin, 1988, p.12). The problem translated into algebraic language becomes an equation of the form $5x + 3 = 40$, with a solution $x = 7.4$. Therefore, in this conception variables are either constants or unknowns. The key instruction here is solve.

**The inequality**

An inequality is an algebraic sentence, which gives a comparison of two or more quantities, in terms of size. It is a statement that a given number or expression is greater or less than the other number or expression. An inequality is also called an inequation.

In this conception, a variable is a pattern generaliser. The conception accepts algebra as generalised arithmetic. Algebra is the study of structures, relations, equality and by substituting concrete numbers, it generalises arithmetic. Algebra deals with generalised numbers or variables and the relationships among these changing quantities. For example, the arithmetic expressions like $2 < 3$ could be generalised to give a property like $x < y$ in inequalities.

Some researchers defined algebra as containing three distinct strands: modelling, functions and generalised arithmetic (Kaput, 2008). Modelling includes solving open number sentences, understanding equivalence and using variables. Functions include processing the ability to recognise, describe, extend and create linear and nonlinear patterns and utilising properties like commutative, associative and the property of zero (Wagner & Kieran, 1989). Generalised arithmetic includes simplifying calculations using number relations.

Identifying what students may learn in algebra is of paramount importance. One formulation of important ideas in algebra is in terms of the habits the students need in algebra (Driscoll, 1999).
Effective algebraic thinking sometimes involves reversibility. It requires the ability to undo mathematical processes as well as do them. It is the capacity not only to use a process to get to a solution, but also to understand a process well enough to work backward from the answer to the starting point.

Furthermore, students should have the capacity for abstracting from computations. This is the ability to think about computations independently of particular numbers used, that is generalising arithmetic. One of the key characteristics of algebra has always been abstractness. Abstracting system regularities from computation is when thinking algebraically involves being able to think about computations freed from the particular numbers in arithmetic.

2.3 Kinds of errors and misconceptions.

A prominent line of research in mathematics education is the study of errors. Students can commit errors due to a myriad of reasons ranging from a data entry to calculation error. Errors are either computational or conceptual in nature. Computational errors like accidentally stating $3x^2 \times 2 = 6x$ or $5x \times 4 = 20$ instead of $6x^2$ and $20x$ respectively require teachers to direct students to their mistakes. However, conceptual errors such as consistently adding instead of multiplying, for example stating that $2x \times 3 = 5x$ or $4x \times 5 = 9x$, instead of $6x$ or $20x$ respectively, require teachers to determine the cause. In this study, an error is a mistake in the process of solving a mathematical problem algorithmically, procedurally or by any other methods Young & O’Shea, (1981).

Computational errors or calculation errors, though seemingly simple to identify, they are not because a teacher has to decide if they are actual calculation inaccuracies or erroneous beliefs. Computational errors must be scrutinised in case they are erroneous beliefs or conceptual errors in masquerade. Erroneous beliefs are the crux behind some of students’ misconceptions. For example if a student does not realise that there is no real square root of a negative number, and present answers as real solutions, then a different instruction intervention is needed.

According to Li (2006) student errors are symptoms of misunderstandings. Among the different types of errors, systematic errors occur to many students over a long time. The cause of systematic errors may relate to student’s procedural knowledge, conceptual knowledge or links between the two types of knowledge.
Orton (1983) classified errors in three categories. There are structural errors from failure to appreciate some relationship involved in solving the problem or to grasp some principle essential to a solution and arbitrary errors in which the student behaves arbitrarily and fails to take into account the constraints in the question. Thirdly, executive errors where the student fails to carry out manipulations through the principle involved that may have been understood.

According to Watanabe (1991), some learners use shortcuts to solve mathematical problems, which results in errors. The student may use shortcut method to simplify $\frac{3x^3-x}{x}$ and writes $3x^3 - 1$ or $3x^2 - x$ as the answer instead of $3x^2 - 1$. The student fails to express $\frac{3x^3-x}{x} = \frac{3x^3 - x}{x}$ or $\frac{x(3x^2-1)}{x}$. Errors can be found in wrongly answered problems, which have flaws in the process that generated the answers (Young & O’Shea, 1981). In trying to solve $2x^2 = x$ the student may commit a cancellation error and writes $\frac{2x^2}{x} = \frac{x}{x'}$, $2x = 1, x = \frac{1}{2}$, instead of $2x^2 - x = 0, x(2x - 1) = 0, x = 0$ or $x = \frac{1}{2}$.

Ricommini (2005), Luneta and Makonye (2010) pointed out that errors and misconceptions are related but they are different and classified errors as systematic and unsystematic errors. Unsystematic errors are unintended, non-recurring wrong answers which learners can readily correct by themselves. According to Lukhele, Murray and Olivier (1999) unsystematic errors are exhibited without the intention of the learners; the learners may not repeat such errors and learners can correct them independently.

Systematic errors though are recurrent wrong responses methodically constructed and produced across space and time. Systematic errors maybe repeated, systematically constructed or reconstructed over a period of time due to grasp of incorrect conceptions of solving a particular problem (Idris, 2011). For example a student repeatedly makes the same error on questions of similar nature; in trying to simplify the expression, the student carelessly leaves out the variable in the final answer. For example to simplify $2x \times 4x$, the student writes 8 instead of $8x^2$ and to simplify $5y - 3y$ the student writes 2 instead of 2y. To simplify $2x + 3x$, the student may mistakenly write 5 instead 5x.

The student systematically commits an error of not writing the variable in the final answer.
Newman (1977) said that when a student attempts to solve a written mathematical problem, then the student has to pass over a number of successive hurdles such as reading (R), comprehension (C), transformation (T), processing (P) and encoding (E). Along the way it is possible to make a careless error. While there are many possible theoretical approaches available for teachers, Newman offers one of the easiest to use and adopt and has proved popular among teachers.

According to Newman (1977), a reading error occurs when a student cannot read a key word or symbol in the written problem. This stops the student from proceeding along an appropriate problem-solving path. A comprehension error occurs when a student can read all the words in the question, but cannot grasp the overall meaning of the words, and is unable to proceed along an appropriate problem-solving path. The transformation error is when the student has understood what the question wants but is unable to identify the operation or sequence of operations needed to solve the problem. A processing skill error is when the student has identified an appropriate operation or sequence of operations but does not know the procedures necessary to carry out these operations accurately. Encoding errors occurs when the student works out a solution to a problem but cannot express this solution in appropriate written form.

2.4 Sources of errors and misconceptions.

One of the main methods used to analyse students’ errors is to classify them into categories based on an analysis students’ behaviour. The causes of student errors are complicated. Students’ errors may be due to carelessness, not understanding at all, confusing different concepts or failing transition from object-oriented thinking to process-oriented thinking.

Generally, an error means a simple lapse of care or concentration, which almost everyone makes occasionally. Students’ errors in using algebra algorithms often are not due to failing to learn a particular idea but from learning or constructing the wrong ideas (Matz, 1980). Errors are signs of misconceptions held by the students. Mathematics ideas and procedures may be correct or may be full of misconceptions.

Ginsburg (1977) offers a number of observations about errors. Ginsberg notes that errors result from original strategies and rules. Faulty rules underlying errors have sensible origins. Too often students see arithmetic as an activity isolated from their ordinary concerns.
Many errors, misconceptions, and faulty thinking in algebra are related or connected to misconceptions and faulty thinking with arithmetic. Resnick (1982) attributed students’ learning difficulties to concept learning. In the domain of algebraic problem solving, one type of prior knowledge that is key to learning is conceptual knowledge of features in the problem such as equal sign, variables, negative sign and like terms. Research proposed that students’ misconceptions or gaps in conceptual knowledge of algebra lead to use of incorrect, buggy procedures for solving problems (Anderson, 1989; Van Lehn & Jones, 1993).

One of the most important findings of mathematics education research is that students constantly invent rules to explain the patterns they see around them (Askew & William, 1995). While these rules may be correct, they may only apply in a limited domain. For example, \( a^x \times b^y = (ab)^{(x+y)} \) is incorrect application of the rule that \( a^x \times a^y = a^{(x+y)} \). Students enter school with self-generated algorithms and problem-solving strategies that represent their priori conceptual understanding of mathematics. Some of these self-generated strategies may be faulty, buggy algorithms, which may generate errors and misconceptions in algebra. Use of incorrect procedures is common when learning algebra (Lerch, 2004; Sebrechts, Enright, Bennet & Martin, 1996), and this inhibits accurate solutions of problems.

When students systematically apply incorrect rules beyond their proper domain of application, we have a misconception. For example, students learn early that a short way of multiplying by ten is to add a zero after the last digit. But what happens to this rule and to the student’s understanding when multiplying ten by fractions or decimals? Askew and William (1995) note that it seems that to teach in a way that avoids students creating misconceptions is not possible and we have to accept that students make some generalisations that are not correct and many of the conceptions will remain hidden unless the teacher makes specific effort to uncover them.

Socas (1997) classified the main causes of errors in learning algebra into two main groups; errors that originate from some obstacle, such as the lack of closure when students see algebraic expressions as incomplete statements, for example \( x + y \), and errors that originate from an absence of meaning. These have two types; complexity of objects and of the processes of algebraic thoughts, for example errors in algebra that have arithmetic origin, errors of procedure and errors in algebra due to the characteristic language and the affective and emotional attitudes of towards algebra.

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Kieran (1992) further elaborated the sources of errors for the use of equal sign. She claimed that students’ tendency to interpret equal sign as a command to compute an answer suggests aspects of arithmetic instruction were contributing to their difficulties in algebra. When students use the equal sign as a step marker to show the next step of the procedure, they do not properly consider the equivalence property of it. One explanation is that the equal sign is an instruction to do something normally comes at the end of the equation and only a number comes after it (Falkner et al., 1999). Another possible source of the misconception is the (=) button on the calculator. Many calculators always return an answer.

Using a cognitive information-processing model and considering specialities of mathematics, Radatz (1979) classified errors in terms of five areas. There are errors emanating from language difficulties. Mathematics is like a foreign language for students who need to know and understand mathematical concepts, symbols and vocabulary. Misunderstanding the semantics mathematics language may cause students’ errors at the beginning of problem solving. Difficulties in processing iconic and visual representations of mathematical knowledge may result in errors. Deficiency in pre-requisite skills, facts and concepts may cause errors; for example, a student may fail to recall related information in solving problems. Incorrect associations or rigidity; that is negative transfer caused by decoding and coding information also results in errors. Finally, application of irrelevant rules or strategies may cause errors.

Misconceptions are derived from problems due to conceptual misunderstanding. One problem that leads to very serious learning difficulties in mathematics is the misconceptions the students may have from previous inadequate learning, informal thinking or poor remembrance. A misconception is a mistaken idea or view resulting in misunderstanding of a concept. It is lack of conceptual understanding. When a student does not fully grasp a concept, they create a framework for a concept that is not accurate. They solve problems through this framework (Bamberger, Oberdorf, Schultz-Ferrell & Leinwand, 2011; Davis, 1995; Hierbert & Carpenter, 1992). When concepts seem to be in conflict with accepted meanings in mathematics, then a misconception has occurred (Osborne & Witrock, 1983).

Radatz (1979) cited by Gunawanden (2011) identified four categories of misconceptions in algebra. These are misconceptions due to processing iconic representations, misconceptions
due to incorrect associations or rigidity of thinking leading to inadequate flexibility in decoding and encoding new information and the exhibition of processing new information, misconception due to deficiencies of mastery of prerequisite skills, facts or concepts and misconceptions due to the application of irrelevant rules or strategies.

Barrera, Medina and Robaynal (2004) categorised misconceptions caused by lack of meaning into three stages: algebra misconceptions originating from arithmetic, use of formula or procedural rule inadequately and misconceptions due to the properties themselves of algebraic language or structural misconceptions.

Nyaumwe et al (2004) found out that some the methods teachers use to teach mathematics do not help students to develop conceptual understanding of mathematics; hence the high failure rate in the subject in Zimbabwe.

2.5 Students’ perspectives on learning and understanding algebra

Despite the significance of mathematics both at school and in life in general, students continue to fail the subject, (Lyons, 2008) and they find algebra difficult to comprehend (Witzel, Mercer & Miller, 2003). Secondary school students experience misconceptions in algebra that their teachers fail to recognise (Smith, diSessa & Roschelle, 1993).

Students are confused and lost when they take algebra. The underpinning of the algebra course is generalisation of the arithmetic they have previously studied and communication about mathematical ideas in a language that is rich in symbols. It is not surprising for students to see algebra as abstract and unconnected to the real world. Mathematics students often find examples, which may appear different but have many common features.

Gunawanden (2011) reported Kuchemann (1981) as classifying students’ interpretation of letters into six categories: letter evaluated, letter ignored, letter as an object, letter as specific unknown, letter as a generalised number and letter as a variable. Students equate learning algebra with learning to manipulate symbolic expressions using transformational rules without reference to meaning of either the expressions or the transformation (English & Halford, 1995).

Most students experience many difficulties if they persist in viewing algebra as generalised arithmetic. Students have cognitive difficulty accepting a procedural operation as part of an answer.
That is, in arithmetic, closure to the statement, 5 + 4 is a response of 9, while in algebra the statement $x + 4$ is a final entity by itself (Booth, 1988, Davis, 1995). In arithmetic word problem, students focus on identifying the operations needed to solve the problem. In algebra word problem, students focus on representing the problem situation with an expression or equation (Kieran, 1990).

Students tend to think every function is linear because of its early predominance in most algebra curricula (Markovits et al., 1988). The implication is that nonlinear functions need to be integrated throughout the students’ experience with algebra.

Having good conceptual knowledge may be necessary for students to solve problems correctly, as deep strategy construction relies on inclusion of sufficient information about the problem features that make them appropriate or inappropriate. Unfortunately, for students with incorrect or incomplete conceptual knowledge about problem features, shallow strategies may prevail.

### 2.6 Related literature

The learning of algebra has received more attention where transition from arithmetic to algebra occurs compared with the goal of arithmetic, which is to find the answer, the focus of algebra is to find a general method and to use algebraic symbols to express these in general forms (Booth, 1988). The reasons for difficulties during transition were investigated from the viewpoint of cognitive development, (Hart, 1981), the use of algebraic notations (MacGregor & Stacey, 1997) and understanding of fundamental concepts like variable, symbolic expressions, equations, inequalities and functions (Usiskin, 1988), which are the building blocks of algebra.

Bases for the research were on initiation to algebra (Coxford & Shuttle, 1988; Socas et al., 1989; Kieran, 1992), specifically on the difficulties and errors with regard to algebraic skills. The literature sources consulted pointed out different aspects of teaching and learning algebra that constituted difficulties and obstacles to learning and which is necessary to deal with.

Buxton (1981) considered the nature of mathematics panic, with algebra being very much part of the mythology of mathematics being very difficult and for selected few. Also critical to algebraic thinking is the capacity to formulate rules to represent functions. This is the capacity to recognise patterns and organise data to represent situations in which input is related to output by well-defined rules like functions.
Variables

The invention of a variable indicates the appearance of modern mathematics (Rajaratnam, 1957). A constant represents only one value but a variable can represent many values. A variable is closely related to the concept of a function. In the equation $y = 2x + 3$, $x$ is the independent variable $y$ is the dependent variable and 3 is a constant and $y$ is a function of $x$. When one variable depends on another for its value, it is a function of another (Philipp, 1999: 157). It can be argued that the most basic aspect of learning of algebra is the fluent use of symbols. In this context, the concept of a variable occupies a prominent position. In many standard algebra texts and the mathematics education literature, one rarely finds an explicit definition of what a variable is. The absence of a precise definition creates a situation where students are asked to understand something, which is largely unexplained, and learning difficulties ensue. Sometimes a variable is described as a quantity that changes or varies. The mathematical meaning of this statement is vague and obscure.

Sometimes it is asserted that student’s understanding of this concept should be beyond that letters can be used to stand for unknown numbers in equations but nothing is said about what it is that should be known beyond this recognition. In NCR (2001) one finds a statement that students emerging from elementary school carry the perception of letters as representing unknowns but not variables (p. 270). The difference between variables and unknowns is unfortunately not clarified. All this adds to the mystery of what a variable really is.

Schoenfeld and Arcavi (1988) argue that understanding the concept of a variable provides the basis for the transition from arithmetic to algebra and is necessary for the meaningful use of all advanced mathematics. The concept of a variable is more sophisticated than teachers expect and it frequently becomes a barrier to a student’s understanding of algebraic ideas (Leitzel, 1989). Some students have a difficult time shifting from a superficial use of $a$ to represent apples to a mnemonic use of $a$ to stand for the number of apples (Wagner and Kieran, 1989).

Expressions

Students treat variables or letters as symbolic replacements for specific numbers. As a result, students expect that $x$ and $y$ cannot both be 2 in the equation $x + y = 4$ or that the expression $x + y + z$ can never have the same value as the expression $x + p + z$ (Booth, 1988).
Algebra students lack the kind of structural conception of algebraic expressions and equations that is necessary for them to use these algebraic objectives as notational tools for proving mathematical relations. Students can formulate correct algebraic generalisations but prefer to confirm the suggested relationships using numerical substitutions. Nonetheless, students do appreciate the more general algebraic demonstrations as part of a proof when someone else performs them (Lee & Wheeler, 1987).

Students over generalise while simplifying expressions, modelling using inappropriate arithmetic and algebra analogies. Using the distributive property, students generate false statements such as $a + (b \times c) = (a + b) \times (a \times c)$, $\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2}$, $(a + b)^2 = a^2 + b^2$ (Matz, 1982; Wagner and Parker, 1993). The role of generalisation is quite important in algebra. However, when introducing a mathematical concept in a correct way, there must be control of the features of such generalisations. Many students try to extend a well-known rule when they do not know specific rules to some of the problems.

Students often can describe a procedure verbally yet not be able to recognise the algebraic representation of this same procedure (Booth, 1984). Students try to force algebraic expressions into equations by adding “$= 0$” when asked to simplify or evaluate (Wagner et al., 1984; Kieran, 1983). They frequently “solve” expressions (Wagner, Rachlin, Jensen 1984) by adding ($= 0$) to expressions they are asked to simplify. Kieran, (1996) affirmed students show absence of knowledge between an equation and an expression. Wagner & Parker (1994) called this the equation – expression problem when the students equate the algebraic expression to zero when asked to simplify. To, simplify $2x + 5 + 3x - 7$, the student did the following: $2x + 5 + 3x - 7 = 0$, $5x - 2 = 0$, $5x = 2$, $x = \frac{2}{5}$.

**Functions**

The definition of a function seems problematic for some teachers and most of the students. Nyikahadzoyi (2006) observed that certain “A” level teachers had difficulties with definition of a function and the same authority noted that some “A” level textbooks called relations like $x^2 + y^2 = 1$ are functions, which is misleading.
Dubinsky and Harel (1992) argue that learners may view a function as a process; a function is taken to be causing change to elements of a set. Nyikahadzoyi (2006) defined learners who can use functions correctly without the concern of correct definition as a structural approach to the function concept. Kyvatinsky and Even (2004) noted that the ability to represent a function in various forms allows the learners to have a deeper understanding of the concept.

The concept of a function is the single most important concept in mathematics education at all grade levels (Harel and Dubinsky, 1992). It emerges from the urge of humans to uncover patterns among quantities, which is as ancient as mathematics (Gagatsis & Shiakhalli, 2004). The concept of function is fundamental and essential in related areas of science. A clear understanding of the function concept is crucial for any student to understand calculus better; a critical direction for the rise of future scientists, engineers, and mathematicians (Carlson & Oehrtman, 2005).

The function concept however, is introduced as a particular type of relation (Jones, 2006). A relation is a generalisation of arithmetic relations, using symbols such as = or < or > that occur in statements such as $2 < 3$ and $2+2 = 4$. A relation is any set of ordered pairs. The set of all first coordinates is the domain; the set of all the second coordinates is the range. An inverse function, however, is a relation when elements in a relation define a link among sets. Thus, a function is a relation in which no two ordered pairs have the same first coordinate.

In Dreyfus’ (1990) summary of the research on students’ work on understanding functions, three problem areas are identified. The mental concept that guides a student when working with a function in a problem tends to differ from both the student’s personal definition of a function and the mathematical definition of a function. Students have trouble with the language of functions for example image, domain, range, pre-images, one–to–one, onto, which subsequently affect their abilities to work with graphical representations of functions (Markovits et al., 1988).

When a function assigns an element $x$ in the domain to a particular element $y$ in the range, we call $y$ the value of the function at $x$, when $y = f(x)$ using the functional notation. The functional notation, $f(x)$, looks awfully like $f$ times $x$ to many students. Teachers should make it explicitly clear that the symbol $f(x)$ is a single unit, with the $x$ value put in the function to generate the value $f(x)$.
The ordered pair \((x; y)\) in which the first coordinate is the member of the domain and the second coordinate is the corresponding value of the function is said to belong to the function. Thus, a function is a set of ordered pairs in which different ordered pairs have different first elements. The zero of a function is any value of \(x\) which satisfies the equation \(f(x) = 0\), in the domain of the function.

Functions are relations only when every input has a different output and named according to their properties. The function concept is related to other concepts like sets, ordered pairs, Cartesian multiplication and relations. Each of these has some specific features and they are presented in forms algebraic and graphical multi-representations. Thus, the concept of a function becomes more difficult (Jones, 2006).

Breidenback (1992) suggested different terms for the function concepts such as pre-function concept, action process and object conception. A student who has developed pre-function concept has little or no idea of a function with elements of one set for this category. An action is a way of dealing with some objects. Learners who view a function as a formula that is used to obtain values of \(x\) have attained the action concept. In this case, a function is taken to be a rule for obtaining a dependent variable from an independent variable.

Wilson et al., (2011) reported misconceptions related to the inverse function, for example writing the inverse of \(y = f(x)\) as \(y = f^{-1}(x)\). This is a common conceptual mistake made in finding the inverse of a function. The reason for such difficulties or misconceptions is that the concept of an inverse function is generally taught based on memorisation and routine rules (Wilson et al., 2011). Such type of teaching approach prevents students from performing the operations in a correct way, understanding how they are used and interpreting them (Baki & Kartal, 2004).

The other reason for these errors is the replacement operation taking place between \(x\) and \(y\) variables since this replacement operation changes the meaning of the variable (Bayazt, 2006; Carlson & Oehrtman, 2005). Using the expression in teaching inverse function concepts or selecting teaching strategies not complying with the subject may cause formation of misconceptions. These are important obstacles in meaningful learning and play a very important role in failing to eliminate errors (Ann & Miroslav, 2009).
It is believed that determining students’ misconceptions and learning difficulties with regard to inverse function will make a significant contribution to the mathematics field, foster understanding and improve performance in finding inverse function and related subjects. Students experience difficulty with functions often because of the different notations. For example, Herscovics (1989) reported that in his research study, 98 percent of the students could evaluate the expression $a + 7$ when $a = 5$ when only 65 percent of this same group could evaluate $f(5)$ when $f(a) = a + 7$.

**Equations**

Equivalent equations have the same root or roots. The equal sign shows equivalence of the expressions. Students lack a good understanding of the concept of equivalent equations. For example, though they can use transformations to solve simple equations like $x + 2 = 5$ becomes $x + 2 - 2 = 5 - 2$), students seem unaware that each transformation produces an equivalent equation (Steinberg et al., 1990).

Although students use the equals sign early in school, they often use it to mean “the answer follows.” For example, in $55 - 43 = ?$, the equals sign can be and often is interpreted as a signal to execute an arithmetic operation (Siegler, 2003). When used in an equation, the equals sign indicates that the expressions on the left and right sides have the same value. This is an obstacle for students who have learned that the equal sign means, “the answer follows.” The additional burden for students is that algebra and arithmetic share the same signs and symbols like the equal sign, addition, subtraction, multiplication and division signs.

Kieran (1992) further elaborated the sources of errors for the use of equal sign. She claimed that students’ tendency to interpret equal sign as a command to compute an answer suggests aspects of arithmetic instruction were contributing to their difficulties in algebra. When students use the equal sign as a step marker to show the next step of the procedure, they do not properly consider the equivalence property of it. One explanation is that the equal sign is an instruction to do something normally comes at the end of the equation and only a number comes after it (Falkner et al., 1999). Another possible source of the misconception is the (=) button on the calculator. Many calculators always return an answer.
Elementary students react in different ways when solving open number sentences involving multiplication and division, for example when solving $6x = 30$, $6 \times _ = 30$, $30 \div _ = 6$, or $30 \div 6 = _$. Open number sentences with the operation on the right side of the equality, for example $30 = _ \times 6$ were significantly more difficult than those with the operation on the left. This perhaps implies that students use different strategies dependent on the problem format or that certain strategies work better with certain problem formats (Grouws & Good, 1977).

When solving equations, teachers consider the transposing of symbols and performing the same operation on both sides to be equivalent techniques. However, students view the two solution processes as being quite distinct. Performing the same operation leads to more understanding perhaps because it visually emphasises the symmetry of the mathematical process. Students using the transportation of symbols technique often work without mathematical understanding and are “blindly applying the Change side-Change Sign rule” (Kieran, 1989).

The interpretation of the equal sign is sometimes different from its accepted meaning. The two meanings attributed to the equal sign are that the symbolic relation show that quantities in both sides are equal and that the transitive relation indicates one quantity on one side can be transferred to the other side using rules. Kieran et al., (1990), in elementary school, an equal sign announces a result. Kieran further claimed that the equal sign is seen by students as “it gives”, that is as left to right directional signal relation; a structural property. Students perceive the equal sign as a symbol inviting them to do something rather than a relationship (Kieran, 1992; Weinberg, 2007; Foster, 2007; Falkner, Levi &Carpenter, 1999). This type of misconception is extensively documented in literature (Kieran et al., 1990; Foster, 2007; Herscovics & Linchevski, 1994). Weinberg (2007) said that instead of uniquely denoting sameness, equal sign seem to be symbols representing ratios, coexistence of unequal sets or an undefined relationship between two objects or ideas or symbols. These various meanings cause students problems.

With regard to understanding equations as object, Kieran (1992) thought that algebraic equations are structural representations that involve a non-arithmetic perspective on the nature of the equations that are depicted. Schoenfield (1987) provided a good example about understanding an equation. If students have difficulty in making a judgement about whether two expressions are equal without computation, such as $(235+122) + (679-122) = 235 + 679,$
such students may understand an equation as a process, that is, the arithmetic approach of computation. On the other hand, students with object-oriented thinking tend to use the property of equation to figure out the unknown value or make a judgement about equality without referring to computation.

**Inequalities**

In real life situations, not all quantities are equal, for example during the distribution of food to the needy people not all families will receive equal shares. Students have encountered the equal sign (=), many times in their mathematics lessons. However, mathematics also caters for quantities that are not equal, (≠), quantities that are greater than others, (>) and some that are less than others, (<).

Inequalities are considered among the most useful tools of investigations in pure and applied mathematics; yet their didactical aspects have not received much attention in mathematics education research. An important aspect of teaching problem solving at secondary level deals with the notion of equivalence of algebraic transformations used in replacing inequalities by equations. Equivalent inequalities have exactly the same solution set.

**Misconceptions**

In the mathematical educational field, research on students’ misconceptions is relatively documented. French (2002) analysed a whole series of misconceptions associated with algebra. Some studies have been undertaken on types of misconceptions students encounter in algebra (An, 2004; Fuchs & Menl, 2009; Jurkovic, 2001) and on the importance of reviewing student errors for increased students’ conceptual understanding (Durkin, 2009). The early research on mathematics learning viewed students’ errors as flaws that interfere with learning and need to be avoided (Greeno, Collins & Resnick, 1996). From an instructional perspective, students’ errors were traditionally perceived either as signals of the inefficiency of a particular sequence of instructions or as a powerful tool to diagnose learning difficulties and direct the related remediation (Ashlock, 1990; Fischbein, 1987).

Students who have an incomplete or incorrect interpretation of the meaning of letters or for the equal sign will have difficulty making sense of this important topic in mathematics. Christou and Vosniadou (2012) noted that, not only do students have a tendency to substitute only
specific numbers for letters; they often limit the specific numbers to natural numbers. For example, in algebra letters stand for specific numbers or as generalised variables.

Research indicates that prior to learning algebra; students must have an understanding of numbers, ratios, proportion, order of operations, equality, algebraic symbolism or letter usage, algebraic equations and functions. These are readiness indicators, which classify prior knowledge necessary for success in algebra (Bottoms, 2003). Lins and Kaput (2004) suggest that the tradition of arithmetic then algebra has contributed to students’ difficulties and recommend an early introduction to algebraic reasoning. Stephens (2008) investigated students’ transition from arithmetic to algebra and the usefulness of students developing rational thinking with number senses to assist with this transition to literal symbols.

Tall (1991) asserted that during a transition from one mental state to another, unstable behaviour is possible with experience of previous ideas conflicting new elements. He emphasised on the terms assimilation where students take in new ideas and accommodation when students’ cognitive structure must be modified to live with new knowledge. This brings up another concept of obstacles.

Cornu, 1991 defines an obstacle as the piece of knowledge of the student, which was at one time satisfactory in solving certain problems. It is precisely the satisfactory aspect which has anchored the concepts in the mind and made an obstacle. The knowledge later proves inadequate when faced with new problems and this inadequacy may not be obvious, thus students make errors.

Research studies identified some pertinent algebraic misconceptions or inconsistencies. Arithmetic and algebra use the same symbols and signs but interpret them differently. For example, an equal sign signifies, “find the answer” in arithmetic and expresses equality between two expressions in algebra (Booth, 1988; Matz, 1982). Arithmetic and algebra use letters differently. Students can confuse the expressions 6 m with 6m, where the first represents 6 metres (Booth, 1988). Arithmetic and algebra treat the juxtaposition of two symbols differently. For example, “8y” denotes a multiplication while “54” denotes the addition 50 + 4. Another example is the students’ inclination that the statement 2x = 24 must imply x = 4. (Chalough and Herscovis, 1988; Matz, 1982). An emphasis on the development of “operation sense” is necessary to prepare students for their introduction to algebraic reasoning.
A suggested approach is the use of word problems plus computational problems as contexts for both constructing and enhancing the meanings for the four basic operations (Schifter, 1999).

Misconceptions occur when students use incorrectly learned strategies to solve new problems (Russell, O'Dwyer & Miranda, 2009). When teachers provide instruction on concepts in mathematics, they are teaching students who already have some pre-instructional knowledge about the topic. Student knowledge, however, can be erroneous, illogical or misinformed. These erroneous understandings are termed alternative conceptions or misconceptions or intuitive theories. These wrong answers are not simple errors but are systematic in nature and come from student past experiences and misunderstandings of such problems (Russel et al., 2009). Misconceptions are incorrect features of student’s knowledge that are repeatable and explicit (Zaslasky & Stein, 1990). For example, secondary school students sometimes think that the longest side of the triangle is the hypotenuse. They think the Pythagorean Theorem applies even when the triangle is not right angled.

Lockhead and Mestre (1988) noted that the research literature consistently indicates that misconceptions are deeply seated in the minds of students and not easily dislodged; in many cases students appear to overcome a misconception only to have the misconception resurface a short time later. This phenomenon is probably because students construct knowledge and become attached to the notions they constructed (Lockhead & Mestre, 1988: 132).

In a misconception framework many students do not come to the classroom as blank slates (Resnick, 1983). Each student brings prior knowledge into the lesson and that knowledge can greatly influence what the student will gain from the experience. Rather, they come with informal theories constructed from their everyday expressions. These theories have been actively constructed. They provide everyday functionality to make sense of the real world but are often incomplete half-truths (Mestre, 1987). They are misconceptions. Misconceptions must be deconstructed and teachers must help students reconstruct correct conceptions. Lockhead and Mestre (1988) describe an effective inductive technique for these purposes. The first step is to probe for and determine qualitative understanding. Then probe for and determine quantitative understanding. Finally probe for and determine conceptual reasoning.

According to Swan (2001: 154), frequently a misconception is not wrong thinking but is a concept in embryo or a local generalisation that the student has made. It may in fact be a natural
stage of development. Although teachers can and should avoid activities and examples that may encourage them, misconceptions cannot simply be avoided (Swan, 2001: 150). Therefore it is important to have strategies for remedying as well as for avoiding misconceptions. Changing the conceptual framework of students is one of the keys in repairing algebra misconceptions. It is not usually useful to merely inform the students on a misconception. The misconception must be changed internally partly through the students’ belief systems and partly through their own cognition.

Misconceptions are one facet of faulty, inaccurate or incorrect thinking or reasoning. They are all inter-twinned causing students unlimited trouble in grasping mathematics from the most elementary concepts to calculus. Students’ misconceptions cause teachers immense frustration about why their teaching is not getting through. Belief systems are resistant to change through traditional instruction (Champagne, Gustone, & Klopfer, 1983; Osborn & Witrock 1983; Confrey & Jere, 1990). Alternative conceptions are not unusual. In fact, they are a normal part of the learning process. We quite naturally form ideas from our everyday experience, but obviously not all the ideas we develop are correct with respect to the most current evidence and scholarship in a given discipline. Moreover, some concepts in different content areas are simply very difficult to grasp. They may be very abstract, counterintuitive or quite complex. Hence, our understanding of them is flawed.

In this way, even adults, including teachers, can sometimes have misconceptions of material (Burgoon, Heddle, & Duran, 2010). Things we have already learned are sometimes unhelpful in learning new concepts or theories. This occurs when the new concept or theory is inconsistent with previously learned material. It is very typical for students and even adults to have misconceptions in different domains or content knowledge areas. Indeed, researchers have found that there are common misconceptions that most students typically exhibit. These alternative theories or misconceptions are very deeply entrenched.

Misconceptions can really impede learning for several reasons. First, students generally are unaware that the knowledge they have is wrong. Moreover, misconceptions can be entrenched in student thinking. In addition, students interpret new experiences through these erroneous understandings, thereby interfering with being able to correctly grasp new information. Misconceptions tend to be very resistant to instruction because learning entails replacing or
radically reorganising student knowledge. Hence, conceptual change has to occur for learning to happen. This puts teachers in the very challenging position of need to bring about significant conceptual change in student knowledge. Generally, ordinary forms of instruction, such as lectures, discovery learning, or reading texts, are not very successful at overcoming student misconceptions. For all these reasons, misconceptions can be hard nuts for teachers to crack.

Research suggests repeating a lesson or making it clearer will not help students who base their reasoning on strongly held misconceptions (Champagne, Klopfer & Gustone, 1983; MacDermott, 1984; Resnick, 1983). Students are emotionally and intellectually attached to their misconceptions partly because they have actually constructed them and partly because they give readily methodologies for solving various problems. They definitely interfere with learning when students use them to interpret new experiences. It is important to recognise the students’ misconceptions and to re-educate students to correct mathematical thinking.

Concerning misconceptions, the challenge is that many students have difficulty relinquishing them. They are deeply ingrained in the mental map of the individual. This is consistent with Hammer, (1996) who identified four properties or basic characteristics of misconceptions. Firstly, misconceptions are strongly held, stable cognitive structures. In addition, they differ from expert conceptions. Thirdly, misconceptions affect in a fundamental sense how students understand natural scientific explanations. Fourthly, they must be overcome, avoided or eliminated for students to achieve expert understanding (p. 99). Ignoring misconceptions may have negative effects on students’ new learning and reinforces original misconceptions. Commonly teachers do not know how to detect the underlying misconceptions in the errors (Ashlock, 2010; Franco, 2008). A misconception is part of the students’ framework that is not mathematically correct or accurate, which leads to providing incorrect answers. Data received from individual students’ work remains individualised and can be cumbersome when helping a class as a whole. The resulting interventions and instructions are generally ineffective for a long-term retention. To correctly identify and efficiently correct, teachers have to discover the underlying patterns of errors that unite every error in a number of different student scenarios. Xiaobao and Yeping (2008) argue that understanding errors is a vital part of correcting them.
2.7 Summary

This chapter discussed the philosophical approach used as a theoretical foundation to study human cognition guiding in this study. It focused on constructivism, its merits and demerits. It highlighted the nature of mathematical understanding in general and algebraic reasoning in particular. Algebra is conceived as generalised arithmetic and the variable is a pattern generaliser. Literature identified the origins or sources of errors and misconceptions. It revealed that errors emanate from the misconceptions the students hold. Researches showed that some errors originate from faulty thinking in arithmetic, faulty rules or buggy algorithms different notations, lack of conceptual knowledge, student invented rules and characteristics of language. Teachers cause pedagogical misconceptions. The methods used do not promote conceptual understanding. Algebra is part of the mythology of mathematics. Students find it difficult and abstract. Many students have difficulty relinquishing misconceptions; they may be deeply ingrained in the mental map of the individual. Misconceptions affect in a fundamental sense how students understand natural scientific explanations. They must be overcome, avoided or eliminated for students to achieve expert understanding. They are also difficult to eradicate or dislodge. Ignoring misconceptions may have negative effects on students’ new learning and may reinforce original misconceptions. Students struggle in mathematics and algebra in particular because generalisation is the underpinning of algebra. The next chapter carries the main methodical constructs of the study, entailing the research paradigm and research design.

3.0 Introduction

The chapter contains the main methodical constructs of the study. It outlines the research paradigm and research design, the population, the sample and sampling techniques, research procedures, the research instruments and their validity and reliability, the pilot study, the main
study, data analysis methods, ethical considerations and a chapter summary. In exploring the nature of algebra, kinds and sources of students’ errors and misconceptions may be illuminated by focussing on five pillars of algebra: variable, equation, function, expression and inequality.

3.1 Research design

The main goal of the study was to identify the nature of algebra, kinds and sources of students’ errors and misconception in this domain. According to Magagula (2004), there are so many paradigms in the field of educational research, ranging from rationalistic, positivism, relational to humanistic methodologies. This study adopted ideas of qualitative paradigm, based on Maxwell’s (2005) five research purposes for qualitative studies. These were to understand the meaning of events, situations, and actions involved, understanding a particular context within which participants act, to identify unanticipated phenomenon to generate new theories, to understand the processes by which events and actions occur and to develop casual explanations. More than one purpose may be achieved in the study.

Selection of a qualitative study was informed Creswell’s (1998) rationale of the nature of the research questions. Normally, qualitative research questions start with how and what questions. The second rationale was to choose a qualitative study when variables could not be easily identified or theories were not available to explain the behaviour of the population. The third one was to choose a qualitative study of individuals in their natural setting. The fourth rationale was to emphasise the researcher’s role as an active learner who could tell the participants’ views rather than as an expert who passed judgement on the participants (p. 18).

McMillan and Schumacher (2010) point out that while most quantitative researchers attempt to establish universal contexts-free generalisations, the qualitative researcher developed context-bound generalisations. The phenomenologist assumed unique case orientation in which the case was special and unique. The first level of enquiry was true, respecting and capturing the details of the individual cases under study.

Based on the assumption that an individuals’ conception of mathematical concepts were dynamic, contextual and could be revealed through in-depth investigation, the research used a case study design. Wiersma and Jurs (2005), a case study is a detailed examination of a specific
event, an organisation or school system. It addressed exploratory, descriptive and explanatory questions. It was a case study “O” level secondary school students learning algebra (Yin, 2008; Johnson & Christensen, 2008). The case study allowed an intensive holistic description and analysis of instruction strategy in a particular mathematics class at one secondary school in the Zimbabwean context. Macmillan and Schumacher (2010), a case study was used because of its flexibility and adaptability to a wide range of contexts, processes and people.

This method, according to Bogdan and Biklen (1992) allowed for prolonged engagement through interaction with participants. Bogdan and Biklen (1992) further point out it was interpretive as it drew its meanings from the participants. The case study was suitable because Lincoln and Guba (1985) claim the findings could be transferred to other similar situations. Yildirin and Simsek (2011), a case study was used to obtain information about the participants and understand the case from various perspectives.

The case study was an in-depth study of instances of a phenomenon in its natural context and from a perspective of the participants involved (Borg and Gall, 1996). It shed light on a phenomenon in process, events or persons of interest to the researcher. The case study was one of the methodologies in a qualitative design (Goldin, 2008). The design helped to examine students’ thinking process, errors and misconceptions in algebra. It provided a unique example of real people in real situations (Cohen, Manion & Morrison, 2005). It further offered a multi-perspective analysis in which the researcher considered not just the voice and perspective of one or two participants in a situation but also the views of other relevant groups and the interaction between them (Nieuwenhuis, 2010).

3.2 Population

A population is the entire group of people from which information of study is extracted (Moore and McCabe, 1998). The population comprised of all form three students at a secondary school in Masvingo district. The school is located 16 kilometres north of Masvingo city. It is a Christian church boarding school and Sunday service is compulsory for all students. The school was selected because these students had the necessary background information in algebra they learnt in form one and form two and the researcher resided there.
Personal demographic data was obtained from the students, (Cross, 2002). The population comprised of 117 girls and 106 boys. Of these 223 students population, only 15 are day-scholars who came from the local community and the rest boarders who got their meals from the dining hall. The day-scholars comprised of 8 girls and 7 boys, provided their own meals. The participants were adolescents of 14 to 17 years of age. The students were conversant in English language. All students had access to compulsory study time in the evenings. The researcher selected form three level because secondary school is a place where students are expected to develop a strong foundation for understanding the algebra concepts that are relevant and necessary in studying mathematics at higher levels.

3.3 Sample and sampling techniques

Borg and Gall (1990) regard a sample as the smaller representative group for study. Systematic random sampling was used. It was quick to carry out and easy to check for errors. Each participant had an equal chance of being selected. The two hundred and twenty three students’ names were alphabetised first. One in every 20 systematic random sampling was done to choose 10 students to participate in the pilot study and these students were later excluded from the main study. One in every three students was sampled to participate in the main study. Seventy-five students were initially selected but some participants withdrew and sixty-five remained; thirty-two girls and thirty-three boys. Chimedza (2003) recommended minimum sample size is 10% of the target population. However, it is not the size of the sample that matters most but its representativeness. Therefore, 29% was fair for this study.

3.4 Data collection procedures

The researcher sought permission to carry out research in schools from the Ministry of Primary and Secondary Education (see Letter attached in Appendix A).

The researcher resided at the station and was familiar with the environment; hence, gaining entry was not difficult. The researcher sought permission to carry out research at school from the headmaster of the selected school and it was granted. The researcher used the form three-class registers and alphabetised all the surnames. The students were systematically random
sampled, taking every one in three student. The 65 sampled students were given a questionnaire to complete (see Appendix E). Each student participant signed a consent form agreeing to take part either in the interview and or in the tests (see Appendix D).

The researcher then carried out a pilot study. Marshall and Rossman (2011:95) define a pilot study as a small-scale preliminary study conducted before the main study. It was a way of testing various features of the study with the sample of the population displaying almost the similar characteristics of target population. The research participants in the pilot study completed the structured questionnaire and wrote a pre-test. The questionnaire (see Appendix E) had two sections A and B. The first part of the questionnaire required personal demographic data of the participants such as student number, gender, age, status (boarder or day scholar), and grade 7 mathematics results. Section B contained 8 True or False statements about students learning and understanding of school algebra. The pilot for the pilot study had 23 question items (Appendix G), on identifying variables, factorising and simplifying expressions, evaluating functions, solving equations and solving inequalities.

The pilot study was a preliminary study to test questions to ensure they made sense to the participants. It helped to assess the feasibility of the research instruments. It also guided the researcher to assess the suitability of the test instrument so that accurate and valid data may be collected in the main study. It assisted the researcher to establish the appropriateness, simplicity, reliability and validity of the instrument. This mini study also helped the researcher to establish and develop a cooperative attitude in the participants. It pointed out the weakness and strengths of both the questions and the sample’s characteristics and alterations were made well before the main study. The pilot study was conducted with ten students in the school.

The researcher marked the students’ written pilot-test and calculated the facility index using the formula: Facility index = \( \frac{n}{N} \) where \( n \) the number of students who answered an item correctly and \( N \) is the total number of students in the sample (McAlpine, 2002). The items that were too easy gave fewer student errors. The difficult items had a low response rate. A facility index between 0.3 and 0.8 was selected for the main study.
Then researcher then administered the main test for the main study to 65 participants. The test contained 25 items. The participants were expected to name a variable and a non-variable, form algebraic expressions from word sentences, quantitative comparison, solving a system of equations in two variables, solving equations and inequalities (see Appendix H).

The researcher interviewed five students whose test scripts showed pernicious errors in algebraic problem solving. The student interview schedule (Appendix F) involved the student reading the question, comprehension, strategy selection, processing (working out questions), explaining procedures, encoding, consolidating and verification.

### 3.6 Research instruments

The researcher explored students’ understanding of algebra, kinds of errors, misconceptions and their causes in learning secondary school algebra. Cohen, Manion and Morrison (2011:34) view research instruments as tools used by the researcher to gather data. They were the means or media used by the researcher to elicit information from the participants. Qualitative methods were used to collect data through student written tests, in-depth student interviews transcripts, researcher’s notes and student written responses to questionnaires (Creswell, 2008). The data collection technique involved the researcher as the chief instrument of data collection.

The researcher followed a qualitative approach by conducting one-to-one interview with students. Interviews with students were worth the time they took as they enabled the researcher to gain insights into the students’ understanding of algebra, procedures solving tasks in algebra and identify error patterns and misconceptions.

**The pilot test**

Chakanyuka (2000) said a test is standardised situation that provides an individual with a score. A pre-test was given first to assess the level of students’ knowledge of secondary school algebra. All the participants passed grade seven mathematics, hence had a basic mathematics background. It was necessary to pre-test the questions to measure reliability and validity. The questions were selected under the five main areas of the study. Features such as overall structure of the test, suitability of items, item coherence, their appropriateness, and validity of the test were discussed with three subject experts to improve validity of the test instruments. Details of the test constructs are in Appendix G.
The main test

The questions in the pilot study and the main study were similar in terms of their structure and difficulty. The test was prepared to obtain data about students’ understanding of the concepts including a variety of items. These test items were directly related to conceptual understanding of algebra. There were six questions on variables, seven questions on expressions, three questions on functions, six questions on equations comprising of linear, quadratic, simultaneous equations and three questions on linear inequalities. Students had to explain basic properties of algebra or they had to identify patterns or relationships and represent them algebraically. Some of them contained manipulations. Problems without a specific context pertaining to simplification of algebraic expressions and solving equations and inequalities, evaluating expressions and functions belonged to this group. For further detail about test constructs refer to Appendix H. Students used mathematical language, algebraic methods to solve problems and justified their answers.

Administration of the main test

After the pilot test, the revised version of the test was prepared with 25 items. The test covered all the five conceptual categories already mentioned. The participants were instructed to use algebraic methods when answering the questions. The duration of the test was one hour but more time was allowed for those who needed it in order to capture more responses. No collaboration among the students was allowed during test session. The researcher personally administered the test to 65 student participants and marked the test. The responses from the test were carefully analysed and grouped according to kinds of errors and misconceptions.

Reliability of the test

A prerequisite for constructing a good test is to ensure reliability. There are several forms of reliability measures in literature. Nunnally, (1992) suggested three methods: alternate-form reliability, retest reliability and split-half reliability. Alternate-form reliability correlates students’ scores by administering two alternate forms of the same test to the same group of students. The retest method gives the same test on two occasions. The split-half aspect needs the same test to given on one occasion only. The test is divided into two parts, normally into odd and even numbered items and the correlation between these two parts are calculated.
There researcher used the split-half method because the alternate form of the test was not available and the retest method is cumbersome. In this method, the test scores were divided into two halves: scores for the odd numbered items and scores for even numbered items. Spearman Brown formula (Gay, Mills & Airasian, 2006) was used to calculate the reliability coefficient of the whole test. 

\[ R_{\text{total test}} = \frac{2r_{\text{split-half}}}{1+r_{\text{split-half}}} \]

The split-half reliability for the pilot study was \( r = 0.42 \) and reliability for the whole test was 

\[ R = \frac{2 \times 0.42}{1+0.42} = 0.59 \]

showing adequate level of reliability. The test was reliable, all the test items correlated with one another (MacAlpine, 2002). Measurements are reliable if they reflect the true aspects but not the chance aspects of what is going to be measured (Gilbert, 1989). The internal consistency of a test was essential for it to serve its purpose.

**Validity of the test**

The validity of the test instrument was as equally important as its reliability. If a test does not serve its intended function then it is not valid. According to Remmers (1965) there are four types of validity namely content, concurrent, predictive and construct validity. Content validity measures how well the content samples the subject matter. Concurrent validity addresses how well the test scores correspond to already accepted measures of performance. Predictive validity deals with how well predictions made from the test are confirmed by subsequent evidence. This type of reliability is not directly relevant to the current study. Construct validity is about what psychological qualities a test measures. This type of validity is primarily used when the other three types are insufficient. In order to preserve validity, the content of the test was discussed with five subject experts and their suggestions were included prior to the administration of the final test. Similar test construct procedures in literature helped when preparing the test items.

**Questionnaire**

Haralambos and Holborn (2008) view a questionnaire as a systematically prepared document with a set of questions designed to elicit responses from the participants for the purpose of collecting data. Questionnaires may be structured or unstructured. The questionnaire is one of the commonly used techniques for data collection (Chimedza, 2003). The researcher used a structured questionnaire. Its advantage was that it was sent to many students at one time.
Participants were assured of anonymity and confidentiality. Cohen, Morrison and Manion (2011) state that in structured questionnaires some control or guidance was given for answers. The questions were short, requiring the participant to provide a yes or no response. The questionnaire provided background information about students’ prior knowledge and attitude towards algebra. Students were also required to indicate their grade seven mathematics result, their gender, age and status; whether day scholar or boarder.

**Student interviews**

Clough and Nutbrown (2012), state that interviews become necessary when researchers feel the need to meet face to face with individuals to interact and generate ideas in a discourse that borders on mutual interest. The researcher elicited oral responses from the interviewee. In line with the constructivist perspective, interviewing was necessary since reflective ability was a major source of knowledge in mathematics. Students articulated their thoughts and verbalised their actions to ensure insights into their thinking processes. During such mental operations, insufficiencies were likely to be spotted.

The researcher used the interview format elucidated by Clements (1980). In this format, questions were classified into input, process and output main areas. In the input phase, the components were reading the problem, interpreting it and selecting a strategy to solve the problem. The process phase involved solving a problem using a selected strategy. The components of the output stage were encoding the verification questions from the interviewer (Clements, 1980). The interviewing used think aloud procedures with a higher version of interview questions which was a dual methodology for the study. The student interview schedule is given in Appendix F. Here are examples of questions in the interview schedule: 1. Please, can you read the question. 2. What does the question mean? 3. How will you solve the question? 4. Workout the question. Explain what you are doing as you proceed. 5. Write down the answer. 6. What does the answer mean? 7. How do you make sure your answer is correct? 8. The interviewer can ask some conflicting questions to prove whether the student has conflict ideas in solving process.

Student face-to-face interviews exposed student reasoning for their misconceptions and errors in the qualitative study. Haralambos and Holborn (2008) say that response given in an interview
may not reflect the real behaviour of the responses because interviews may threaten the participants’ ability to express themselves. Participants may be nervous about the researcher’s presence and give false information. In this regard the researcher first established good rapport with the interviewees during the booking stages.

**Selection of students for interviews**

Interview participants were selected by thoroughly examining the answers in the test. Students who displayed serious, pernicious errors and misconceptions were selected for interviews. At the end of the test 5 students were selected for interviews. The students were selected purposively to represent the ability levels of whole class. The students were asked to explain their answers to elaborate errors and misconceptions. The interviews allowed for the exploration of students’ algebraic reasoning abilities. All of these interviews were audio taped and transcribed. Each interview lasted approximately 15-30 minutes.

**3.8 Data presentation and analysis procedures**

Since the aim of the study as to explore students’ understanding of algebra and identify student errors underlying misconceptions, correct answers and no responses were eliminated in the presentation and analysis of data. Rubric of error categories were tabulated as per conceptual area. Literature on conceptions of algebra, types and sources of errors and misconceptions guided themes or error categories. Each question was scored correct [1] and wrong [0]. A response was considered correct if the answer came from a complete correct algebraic method or procedure. Wrong answers were compiled and grouped according to type of error or misconception. The research was open ended to include other categories of errors and misconceptions. The researcher analytically identified errors for other categories not in literature. Data was presented as per research question.

Data analysis and interpretation were from a phenomenological perspective or from the questions obtaining. The study adopted content analysis approach with major focus on students’ written responses to questionnaires, tests and interview transcripts. It entailed critical examination rather than mere description of students’ responses in the written test, questionnaire and interview transcripts. It is an analytical method used in qualitative research to gain understanding of trends and patterns that emerge from data (Daymon & Holloway, 2011). Content analysis helped to discover new or emergent patterns including overlooked
categories (Daymon & Holloway, 2011, p.321). It supplemented the data from other methods (Bell, 1995). Identification of common errors through examination of completed student work was important because once a student’s errors were isolated; the teacher could direct corrective instruction or a remedial plan aimed at that particular error pattern (Riccomini, 2005).

3.9 Ethical considerations

Styron (2001, p. 23), argues that every research should be guided by ethical underpinnings like seeking consent from the participants, no deception on the participants, no violation of participants’ privacy and release and publication of findings in an accurate and responsible manner. The researcher sought ethical clearance and permission from the Ministry of Primary and Secondary Education to access the school and conduct the study. The researcher also sought permission from the Headmaster of the school. Each research participant signed a consent form to participate in the research. The researcher informed the participants why they were selected to participate in the research. The researcher furnished participants with details of the research and the potential risks, objectives, intentions and benefits of the study. The research participants voluntarily participated. They were not coerced to participate in the research. They were free to withdraw their participation at any stage of the research process without any prejudice. The participants were assured that all data collected would be kept strictly confidential and would be only accessible to the research assistants. Data collected will be for at least five years after completion, presentation and or publishing of the study and then it will be destroyed. Questionnaires were only identified by pseudo-names to ensure anonymity. Information collected was correctly acknowledged to the sources. The researcher would be honest about the nature of the findings even if they tended to contract the main thrust of the argument. The principles of respect for personal autonomy, benevolence and justice guided the research. Sensitivity to ethical considerations was upheld throughout the study.

3.10 Summary

The researcher used a qualitative research paradigm. A case study was the main design of the qualitative approach. The section discussed the population for the study. It looked at sample and systematic random sampling as a sampling procedure. Data collection procedures were discussed. They included seeking permission from relevant authorities to conduct research school, signing of participant consent forms, carrying out a pilot study and the main study,
administering a questionnaire, pilot test and the main test. The main research instruments were tests, structured questionnaire and student interviews. Student interviews served the qualitative purpose of the research instrument. Data collected from students’ written responses to questionnaires, written tests and interviews were presented in tabular form, as per research question. Students’ answers to the test, their written work, interview transcripts and the researcher’s field notes constituted content for content analysis of data. Content analysis was mainly explanatory and a narration of findings. Finally yet importantly, ethical considerations were also tackled in this chapter. The next chapter focuses on data presentation, data analysis and discussion of research findings.

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Chapter Four

Data Presentation, Analysis and Discussion

4.0. Introduction.

This chapter focuses on the findings of case study research design. It presents findings as per research question of the study. The chapter discusses findings from a structured questionnaire, two written tests and students’ interviews. Focus was mainly on students’ conceptions of algebra, the kinds of errors, misconceptions and their origins. Data was presented mainly in form tables for easy interpretation. The twenty-five test items were classified into one of the five conceptual areas; variables, expressions, functions, equations and inequalities. The errors and possible misconceptions in each question item were noted and put into various categories. Since the aim of the study was to explore students’ understanding of algebra and identify student errors and underlying misconceptions, data presentation and analysis did not include correct answers and non-responses.

4.1 Research question 1. How do secondary school students conceive algebra?
Regarding students’ conception of algebra, students were given a questionnaire. The questionnaire responses revealed that some students were aware that a letter can represent any number in algebra. The students’ understanding of a variable was consistent with the formal definition of a variable as pattern generalise (Booth, 1986; Usiskin, 1988).

Table 1. Student questionnaire extract

<table>
<thead>
<tr>
<th>STATEMENT</th>
<th>TRUE</th>
<th>FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. A letter can represent any number</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. The four operations are applicable to arithmetic only</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Rules of precedence are applicable to algebra</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Understanding arithmetic is key to understanding algebra</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Algebra is one of the most important areas of mathematics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Algebra generalises arithmetic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Algebra is easy to understand</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. Algebra is interesting to learn</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

However, from scanned response in the questionnaire above, some students had a limited understanding of the connection between arithmetic and algebra. They were not aware that rules of precedence are also applicable to algebra. As revealed from literature and the scanned responses above, students’ general perception was that algebra is a difficult domain. This is confirmatory to Greens and Rubenstein (2008) that students still struggle with algebraic concepts. It is consistent with Buxton (1981); nature of mathematics panic and algebra being part of that mythology of being very difficult and for the selected few. For example, this student could not realise that understanding arithmetic is key to understanding of algebra. In the pilot test though students had a limited understanding of a variable, some students had something close to the notion as evident in student’s scanned response to definition of a variable. The student accepted the conception of algebra when letters are used as arbitrary objects in algebraic expressions. This is consistent with Gunawaden’s (2011) assertion that a variable is difficult to define because its definition depends on the context.

1. What is a variable? Explain with an example.

Answer: An algebraic letter which appears on different algebraic expressions.
Students’ errors and misconceptions related to variables

There were five questions in the main test which asked for students’ understanding the notion of a variable. These were questions 1, 2, 3 and 5(b). There were different forms of responses. The responses showed complete understanding, partial understanding or lack of understanding of the notion of a variable. Sometimes there were no visible reasons for these incorrect answers. The type of algebraic misconceptions included assigning labels for variables, assigning labels for constants, misinterpreting product of two variables, misjudging magnitudes of variables, lack of understanding of variables as generalised numbers and forming wrong equations.

Question 1

Bene sells $x$ sweets and Billy sells twice as many sweets as Bene. A sweet costs 10 cents. (a) Name a variable in this problem. (b) Name something that is not a variable in this problem.

Question 1 was meant to test if students had proper understanding of the concept of variables. Some students misinterpreted a variable as a label or as a thing or a verb.

They really failed to perceive the correct interpretation of the variable as the number of a thing. Some students had difficulties in distinguishing between variable and non-variables in terms of varying and no-varying quantities in the question. This is consistent with Schoenfeld and Arcavi (1988) propounding that understanding the concept of a variable is key for transition from arithmetic to algebra. Also some students formed equations when they were not necessary. When students were asked to name a variable and a non-variable, some students gave answers in form of equations. There was no meaning attached to these equations and they revealed a false relationship between the variables and constants in the question. It is very difficult to predict any theoretical attachment of the answer with the question.
Some students were could not view variables as constants or vice versa. This error was evident when students were asked to name something not a variable; the responses like Billy and Bene, sweets were given. Generally, these answers may be considered as correct. Sometimes words like sweets or cents could be regarded as symbols representing variables in some contexts. However, these answers were incorrect in the context of the given question since was variable or a given number attached to these words. Therefore, these words had meanings attached to the give context when taken together with these variables or numbers. The responses to this item showed complete lack of understanding of the notion of a variable in algebra by some students. This confirms Leitzel’s (1989) assertion that the concept of a variable is more sophisticated and a barrier to understanding algebra. For example question 2: What does $xy$ mean? Write you answer in words. The students who had responses like expressions, unknown; coefficients or variables could have completely misread the question. They had complete lack of understanding of a product of two variables

One of the students wrote $by$ $y$. This was ambiguous and the student was interviewed for further clarification: R stands for researcher. S stands for student.

R: What do you mean when you write $x$ by $y$?

S: I divide $x$ by $y$?

The student completely lacked conceptual understanding of product of two variables.

Question 3 tested students their conception of variable in algebra. It read:

2. Bene sells $x$ sweets and Billy sells twice as many sweets as Bene.

(a). Name a variable in this problem.

(b). Name another variable in this problem.
Apples cost $a$ cents and bananas cost $b$ cents. If 3 apples and 2 bananas are sold, what does $3a + 2b$ represent?

Some students displayed lack of understanding of the unitary concept when dealing with variables. Their area of difficulty was in understanding the unitary concept when multiplying a constant with a variable; when the cost of apples is $a$ has to be multiplied by 3 or price of bananas $b$ has to be multiplied by 2. This is a basic arithmetic concept. The prices were given as variables. Students who had this misconception perceived the $3a$ as 3 apples for $a$ cents and $2b$ as 2 bananas for $b$ cents as in the scanned solution.

It is evident that, in addition to the incorrect calculation, they regarded $a$ as the label for apples and $b$ as the label for bananas, rather than the unit price of an apple and the unit price of a banana and regarded $a$ and $b$ as prices of item. The concept of a variable is more sophisticated than teachers expect and it frequently becomes a barrier to a student’s understanding of algebraic ideas (Leitzel, 1989). For example, some students have a difficult time shifting from a superficial use of $a$ to represent apples to a mnemonic use of $a$ to stand for the number of apples (Wagner and Kieran, 1989). The student misunderstood a variable in algebraic expressions as an arbitrary object as shown by the scanned solution below.

Maybe the student thought the question required operations to be performed, clearly showing lack of conceptual understanding concept of a variable.

**Question 5(a)** Which is larger $\frac{1}{n}$ or $\frac{1}{n+1}$, when $n$ is a natural number? The following is an excerpt of the interview.

R: Could you please read the question for me?

S: Which is larger, one over $n$ or one over $n$ plus one?
R: What is your answer?

S: $\frac{1}{n+1}$ is greater than $\frac{1}{n}$.

R: Could please explain?

S: I put 2 in $n + 1$ so I get 3 and in $n$ I have 2 so 3 is greater than 2.

R: Which is greater $\frac{1}{2+1}$ or $\frac{1}{2}$?

S: $\frac{1}{2}$ is greater than $\frac{1}{3}$.

R: What values can $n$ take?

S: Um...I’m not sure.

Table 2: Errors and misconceptions in variables

<table>
<thead>
<tr>
<th>Conceptual error</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incorrect quantitative comparison</td>
<td>$\frac{1}{n}$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
</tbody>
</table>

Question 5(a) was meant to find out students’ proper understanding quantitative comparison.

A high frequency of errors was attributed to incorrect quantitative comparison in comparing two algebraic fractions. These students substituted numbers to the algebraic expressions in order to compare them. After the substitution, they only compared the magnitudes denominators, instead of comparing the whole fractions thereby arriving at faulty conclusion that $\frac{1}{n+1} > \frac{1}{n}$. Students prefer to confirm relationships by numerical substitutions (Lee & Wheeler, 1987). They failed to realise that the reciprocal of a number is smaller than the number
itself under certain conditions. Such students evidently compared only the magnitudes of denominators rather than comparing the magnitudes of the whole fractions. She confessed lack conceptual knowledge of natural numbers represented by a variable. The variable was again misconstrued in question 5. The student seemed to have problematic understanding of quantitative comparison when comparing two algebraic fractions. The scanned solution reflected lack of conception of a variable.

**Question 5 (b)** Which is greater $y$ or $x$ in $y = 2x + 3$?

**Table 3: Students’ errors and misconceptions in Question 5 (b)**

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lack of understanding of variable as a generalised number (conceptual error)</td>
<td>$x$</td>
</tr>
<tr>
<td></td>
<td>$y$</td>
</tr>
<tr>
<td>Misjudging the magnitudes of variables (conceptual error)</td>
<td>They are equal</td>
</tr>
<tr>
<td>Misread the question: made $x$ subject of formula (Reading error)</td>
<td>$x = \frac{y - 3}{2}$</td>
</tr>
</tbody>
</table>

This question tested students understanding of a variable as a generalised number. This conceptual knowledge was lacking resulting in conceptual errors like stating $y > x$ or $x > y$ or $x = y$. They could not realise that they are two different variables, either one could larger depending on the number. Another conceptual error was misjudging the magnitude of the two variables. The students who misread the question committed a reading error according to Newman (1977). It was also meant to test students’ ability to judge the magnitudes of variables and their proper understanding of variables as generalised numbers. Some students did not realise that a variable can take many values in the same context was a problem in an equation such as $y = 2x$. These students partially understood that $x$ and $y$ are variables. However, with limited conceptual understanding, they failed to perceive that these variables can assume more than one value. Moreover, they focussed only on the domain of positive real numbers when substituting values for $x$ and $y$ to find which is larger. It was necessary to substitute values from the negative domain and this would yield a different result altogether. Students treat variables or letters as symbolic replacements for specific numbers (Booth, 1988).
Another misconception here was that some students considered \( y \) as the number obtained by doing some operations to the right hand side of the equation. In fact, they perceived the equal sign as “to do something” to the right hand side of the equation to get the answer on the left hand side (Kieran, 1992; Falkner).

4.2 Research question 2.

What kinds of errors and misconceptions do students experience in learning school algebra?

Students written tests manifested various taxonomies of errors and misconceptions. The broad categories of errors noted were conceptual errors and computational or procedural errors. Conceptual errors emanate from student’s inadequate knowledge of concepts. Computational errors are calculation errors. Procedural errors are a result of a wrong or incorrect method in the process of solving a problem.

**Question 1** Bene sells \( x \) sweets and Billy sells twice as many sweets as Bene. A sweet costs 10 cents. (a) Name a variable in this problem. (b) Name something that is not a variable here?

**Table 4: Errors and misconception in variables**

<table>
<thead>
<tr>
<th>Conceptual errors</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assigning labels for variables</td>
<td>( x ) or ( 2x )</td>
<td>Sweets, 2times, twice</td>
</tr>
<tr>
<td>Assigning values for variables</td>
<td>( x ) or ( 2x )</td>
<td>10cents</td>
</tr>
<tr>
<td>Assigning verbs for variables</td>
<td>( x ) or ( 2x )</td>
<td>Sold or sells</td>
</tr>
<tr>
<td>Assigning constants for variables</td>
<td>( x ) or ( 2x )</td>
<td>Billy or Bene</td>
</tr>
<tr>
<td>Assigning labels for constants</td>
<td>10 cents or 10</td>
<td>Billy or Bene, cents, sweets</td>
</tr>
<tr>
<td>Assigning variables for constants</td>
<td>10 cents or 10</td>
<td>( x ) or ( 2x )</td>
</tr>
<tr>
<td>Assigning verbs for constants</td>
<td>10 cents or 10</td>
<td>Sell or sold</td>
</tr>
<tr>
<td>Forming wrong equations</td>
<td>( x + 2x = 10 )</td>
<td>( x = 2x ) or ( x = 10 ) or ( 2x = 10 )</td>
</tr>
</tbody>
</table>

For example in question 1 of the main test students were asked to identify a variable and a non-variable. The conceptual errors were in form of assigning labels, for variables (sweets, 2times, twice), assigning values(10 cents), assigning verbs for variables (sold or sell), assigning constants for variables (Billy or Bene), assigning labels for constants (Billy or Bene, cents, sweets), among other errors in the table above. Students lacked conceptual knowledge of a variable for example in the scanned solution.
Question 2: What does $xy$ mean? Write you answer in words.

Table 5: Students’ errors and misconceptions on algebraic expressions

<table>
<thead>
<tr>
<th>Conceptual error</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misinterpreting the product of two variables</td>
<td>$x$ multiplied by $y$ or $x$ times $y$</td>
<td>$x \times y$</td>
</tr>
</tbody>
</table>

In question 2 conceptual errors were noted. Some students had no conceptual understanding of the product of two variables and wrote among the incorrect responses expression, unknown and coefficients. Some students had partial conceptual knowledge as they were able to know that $x$ and $y$ are variables.

Question 3 tested students’ conception of variable in algebra. It read: Apples cost $a$ cents and bananas cost $b$ cents. If 3 apples and 2 bananas are sold, what does $3a + 2b$ represent?

Table 6. Students’ errors and misconceptions on the idea of a variable

<table>
<thead>
<tr>
<th>Conceptual errors</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lack of understanding of the unitary method when dealing with variables</td>
<td>The total cost 3 apples and 2 bananas</td>
<td>$3a + 2b$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3cents and 2cents</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$5ab$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Expression</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Price of apples and bananas</td>
</tr>
<tr>
<td>Variable as a label</td>
<td>The total cost 3 apples and 2 bananas</td>
<td>3 apples and 2 bananas</td>
</tr>
</tbody>
</table>

The student made a conceptual error because of lack of understanding of the unitary concept. Some had $5ab$ which was a serious misconception of adding unlike terms.
3. Apples cost $a$ cents and bananas cost $b$ cents. If 3 apples and 2 bananas are sold, what does $3a + 2b$ represent?

Answer: $3a + 2b$

Students’ errors and misconceptions related to algebraic expressions.

There are 8 questions on expressions in the test. These were questions 4(a), (b),(c), 5(a), 6(a), (b), (c) and (d). The errors were mainly procedural in nature as they were a result of error in trying to solve a problem. These were invalid distribution, oversimplification, incomplete simplification, misreading the question, reversal error, invalid distribution, illegal cancellation and interference from previous methods. It can be pointed out that students had serious problems in simplification of algebraic expressions.

**Question 4** (a) Multiply $x + 3$ by 2  (b) Add $4x$ to 3  (c) Subtract $2x$ from 7

**Table 7: Errors and misconceptions in algebraic expressions**

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invalid distribution</td>
<td>4(a) $2x + 6$</td>
<td>$x + 6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x + 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x + (3 \times 2)$</td>
</tr>
<tr>
<td>Oversimplification</td>
<td>4(a) $2x + 6$</td>
<td>$2x + 6 = 8x$ or 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$2x + 3 = 5$ or $5x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x + 6 = 7x$</td>
</tr>
<tr>
<td></td>
<td>4(b) $4y + 3$</td>
<td>$7y$</td>
</tr>
<tr>
<td></td>
<td>4(c) $7 - 2b$</td>
<td>$7 - 2b = 5b$</td>
</tr>
<tr>
<td>Incomplete simplification</td>
<td>4(a) $2x + 6$</td>
<td>$(x + 3)2$</td>
</tr>
<tr>
<td>Invalid equation</td>
<td>4(b) $4y + 3$</td>
<td>$4y + 3 = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y = -\frac{3}{4}$</td>
</tr>
<tr>
<td>Misread question</td>
<td>4(a) $2x + 6$</td>
<td>$2(x + 2) = 2x + 4$</td>
</tr>
<tr>
<td>Reversal error</td>
<td>4(c) $7 - 2b$</td>
<td>$2b - 7$</td>
</tr>
</tbody>
</table>

Question 4(a) was designed to test students their proper understanding of multiplication of algebraic expression by a number. It was a word problem to be changed to an algebraic expression. Invalid distribution was observed in this question. This is a kind of misuse of the distributive property in algebra. The distributive property states that $x(y + z) = xy + yz$. It implies either addition first and then multiplication or multiplication first then addition. It
makes no difference. However, when unlike terms are inside the brackets it is impossible to add them. Students multiplied terms inside the brackets by the letter outside of the parentheses. Actually the distributive law helps us to simplify algebraic quantities by allowing us to replace terms containing parentheses with equivalent terms without the parentheses anymore.

Many forms of incorrect use of the distributive property were found under invalid distribution. One such category of misuse of this property was incomplete distribution. Sometimes students began to apply the distributive property but failed to complete the process leaving incorrect answers such as \(2(x + 3) = 2x + 3\) or \(2(x + 3) = x + 6\) or \(2(x + 3)\). From the pilot test incomplete scanned solution is presented. The student had partial understanding of factorisation and limited knowledge of factors of difference of two squares inside the parentheses.

Another error or misconception occurred on this question when some students oversimplified their answers after correct expansion. This was the largest category of errors on this question. The most interesting feature of this category was that students conjoin, connect or even put together terms without even considering the operations that are to be carried out on these terms, giving such answers as \(2x + 6 = 8x\). There were also two cases of misreading the question where students wrote \(2(2x + 2) = 4x + 4\), thus a reading error occurred.

Question 4(b) was a word problem to test students’ understanding of forming of algebraic expressions from word sentences. Students formed the, correct expression. However, like in 4(a) oversimplification featured again, with incorrect answer \(4x + 3 = 7x\). Also a case of invalid equations was observed on this question when students formed equations where they are necessary. The student went on to solve forged equation, \(4x + 3 = 0\), \(x = -\frac{3}{4}\). From the pilot test formed invalid equations instead of simplifying expressions.
4. Simplify the following expressions:

(a). 3(x − 5)

(a). Answer: 3x − 15

Question 4(c) tested students again their proper understanding of algebraic expressions in word problems. Students were asked to subtract 2x from 7. Incorrect word matching led to reversal error when forming algebraic expressions from sentences. When the subtrahend was an algebraic term and minuend was a number in the word sentence, some students carried out the operation in a reversed order by exactly matching the letters in the given order. The common wrong answer was 2x − 7. Oversimplification error was noted on this question, despite having formed the correct expression 7 − 2x, some students incorrectly presented 7 − 2x = 5x.

Table 8: Students’ errors and misconceptions in algebraic expressions

<table>
<thead>
<tr>
<th>Procedural errors</th>
<th>Expected response</th>
<th>Students’ incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oversimplification</td>
<td>6(a) 7 + 3x</td>
<td>10x</td>
</tr>
<tr>
<td>Forming invalid equation</td>
<td>6(a) 7 + 3x</td>
<td>x = 7/3 or 7/3</td>
</tr>
<tr>
<td>Oversimplification</td>
<td>6(b) -3p − 2c</td>
<td>-5pc or 6pc</td>
</tr>
<tr>
<td>Invalid distribution</td>
<td>6(b) -3p − 2c</td>
<td>7p − 2c, −10p^2 -4pc, 3p − 2c, 2cp + 5p^2</td>
</tr>
<tr>
<td>Incomplete simplification</td>
<td>6(b) -3p − 2c</td>
<td>p + p − 2c − 5p, 2p − 2c − 3p</td>
</tr>
<tr>
<td>Incorrect cross multiplication</td>
<td>6(c) ( \frac{ax}{b} ) or ( \frac{bx}{a} )</td>
<td>( \frac{ax}{bx} ) or ( x^{a-b} )</td>
</tr>
<tr>
<td>Interference from previous methods</td>
<td>6(c) ( \frac{ax}{b} ) or ( \frac{bx}{a} )</td>
<td>( \sqrt{xa} ) or ( x^{a-b} )</td>
</tr>
<tr>
<td>Oversimplification</td>
<td>6(d) ( \frac{a+b}{1+d} )</td>
<td>( \frac{a+b}{1+d} ), ( a + \frac{b}{d} )</td>
</tr>
<tr>
<td>Incomplete simplification</td>
<td>6(d) ( \frac{a+b}{1+d} )</td>
<td>( \frac{x(a+b)}{x(1+d)} ), ( x(a+b) ÷ x(1+d) )</td>
</tr>
<tr>
<td>Illegal cancellation</td>
<td>6(d) ( \frac{a+b}{1+d} )</td>
<td>( \frac{a+b}{1+d} ), ( \frac{a+b}{x} ÷ \frac{1+d}{x} )</td>
</tr>
</tbody>
</table>

Question 6(a) required students to simplify 7 + 3x. One student failed to realise that an algebraic expression can be a complete final answer and oversimplified to get incorrect answer.
7 + 3x = 10x. This is consistent with Socas (1997) who suggests lack of closure and absence of meaning. Another student committed incomplete simplification. An answer is classified as incomplete when some students terminated the simplification of the algebraic expression somewhere in the middle of the process without realising the final answer. In the student’s perception, these answers are final and complete but they are incomplete when compared to the standard algebraic procedures. Another possibility is that these students may not know how to proceed further. Some student wrote the problem again in another format as the answer, $7 + 3x = 7 + 3 \times x$ or they terminated the procedure without completion.

Another error was forming invalid equations possibly emanating from misreading the question, yielding $7 + 3x = 0$, $x = \frac{-7}{3}$ and incorrectly $x = \frac{7}{3}$. This confirmed Wagner and Parker’s (1984) equation-expression problem when students force expressions into equations and solve instead of simplifying and evaluating.

Question 6(b) tested students’ simplification of algebraic expressions involving brackets. Students were instructed to simplify $x + x - (2x + 5x)$. Various forms of wrong answers were realised from oversimplification after obtaining correct answer. Students did not know when to stop the simplification. Another major error was also invalid distribution which was aggravated by presence of negative sign before the bracket. Also incomplete simplification was observed in spite of having correctly removed the parentheses.

Question 6(c) tested students’ multiplication of an algebraic fractional expression by a letter. Students were asked to simplify $x \left( \frac{a}{b} \right)$. The major errors observed were oversimplification with incorrect answer $\frac{a}{b}$. Some incorrect cross multiplication was noted. When students multiplied an algebraic fraction by a letter, they often multiplied both numerator and denominator of the fraction by the letter to get $\frac{ax}{bx}$. Sometimes they may assume that there is no denominator to the letter. Often this happens when there is no visible denominator. They seem to have difficulties in realising that a single letter can be represented by an algebraic fraction by taking the denominator as 1. Because of this lack of understanding, students tend to assume that both numerator and denominator of the fraction should be multiplied by the letter. There were also errors emanating from previous learnt methods, when some students misconstrued
this question for a question involving indices or exponents. These student wrote such incorrect answers as $\sqrt[n]{x}$ or $x^{a-b}$.

On question 6(d) students were required to simplify $\frac{ax+xb}{x+xd}$. This question tested students’ proper understanding of simplification of algebraic fraction by factorisation. One error noted was oversimplification, resulting in incorrect answers like $\frac{a+b}{d}$ and $a + \frac{b}{d}$. Another category of errors on this question was incomplete simplification, when students correctly factorised out $x$ in both numerator and denominator but fail to divide denominator and numerator by $x$. Incomplete answers like $\frac{x(a+b)}{x(1+d)}$ and $x(a + b) ÷ x(1 + d)$ featured. The scanned script is evidence of incomplete solutions.

The other category was illegal cancellation. Some students invented shortcuts when they just crossed out $x$s without going through the correct procedure of factorisation (Young & O’Shea, 1981). They obtained incorrect answers like $\frac{x(a+b)}{x(1+d)} = \frac{a+b}{d}$.

**Students’ errors and misconceptions related to functions.**

The whole of question 8 was based on functions. Students were given that $f(x) = x^2 - 1$. In question 8(a) students were asked to find $f(0)$ and this question tested proper understanding of the functional notation and correct substitution. They were required to evaluate the given function when $x = 0$. Incomplete simplification was one of the errors that were noted after correct substitution. Students got stuck and could not proceed any further after writing $0^2 - 1$ or 0-1. They thought may be that was the final answer or maybe it was a reflection of a fragile understanding of directed numbers. There was also a clear lack of understanding of the functional notation when students presented answers of the form $f(0) = x^2 - 1 = 0$. Question 8(b) like the previous one required proper understanding of correct substitution of a negative number to find $f(-2)$. The students who chose to disregard the correct use of essential brackets perished on this item. Incomplete simplification, $(-2)^2 - 1$ or 4 - 1 and lack of understanding.
of the functional notation, \( f(-2) = x^2 - 1 = -2 \) were quite evident on this question. This agrees with Nyikahadzoyi (2006) who propounds that the definition of a function seems problematic for some “A” level teachers and some students.

On question 8(c) students were required to solve the function for \( x \) when \( f(x) = 0 \). Lack of understanding of the functional notation was observed when students gave answers like \( f(x^2 - 1) = 0 \). Jones (2006) and Markovits et al. (1989) concur that students have trouble with language of functions. Incomplete simplification also featured when students presented \((x - 1)(x + 1) = 0 \) and \( x = \pm \sqrt{1} \). Some students failed to identify quadratic nature of this question. Also there was interference from previous learnt methods in solving equations. The misconception here was that some students thought that to solve an equation they must divide both sides of the equation by something. Students wrote \( x^2 - 1 = 0 \) then incorrectly had \( x = \frac{0}{x^2 - 1} \). Guessing was also evident when presented answers only without any evidence of some working. Answers like 0, 1 or \( x^2 \) just appeared in the answer space. The findings support Herscovics (1989) who asserts that students experience difficulties with functional notations.

**Table 9: Students’ errors and misconceptions in question 8**

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incomplete simplification</td>
<td>8(a) -1</td>
<td>0^2 - 1; 0 -1</td>
</tr>
<tr>
<td></td>
<td>8(b) 3</td>
<td>(-2)^2 - 1; 4 -1</td>
</tr>
<tr>
<td></td>
<td>8(c) 1 or -1</td>
<td>((x - 1)(x + 1) = 0) (x = \pm \sqrt{1})</td>
</tr>
<tr>
<td>Lack of knowledge of the functional notation</td>
<td>8(a) -1</td>
<td>(f(0) = x^2 - 1 = 0)</td>
</tr>
<tr>
<td></td>
<td>8(b) 3</td>
<td>(f(-2) = x^2 - 1 = -2)</td>
</tr>
<tr>
<td></td>
<td>8(c) 1 or -1</td>
<td>(f(x^2 - 1) = 0)</td>
</tr>
<tr>
<td>Disregard use of brackets</td>
<td>8(b) 3</td>
<td>(-2^2 - 1 ) or (4 - 1 = -5)</td>
</tr>
<tr>
<td>Interference from previous methods</td>
<td>8 (c) 1 or -1</td>
<td>(x^2 - 1 = 0) (\text{then } \frac{0}{x^2 - 1})</td>
</tr>
<tr>
<td>Guessing</td>
<td>8(c) 1 or -1</td>
<td>1; 0; (x^2)</td>
</tr>
</tbody>
</table>
Some students displayed clear lack of understanding of the functional notation. Their main challenge was interpreting the functional notation. The functional language or semantics was misunderstood by most of the students (Radatz, 1979). The error categories were a combination of conceptual (disregarding use of brackets and interference from previous methods) and procedural included incomplete simplification. Some errors also emanated from guessing and lack of understanding of the functional notation.

**Students’ errors and misconceptions related to algebraic equations**

There were six questions in the main test on algebraic equations. These were questions 7(a), (b), (c), 9(a), (b) and (c). The major errors were procedural including misinterpreting the elimination method, incomplete solutions, misuse of the change side–change sign rule, disregarding the use of brackets, disregarding the use of the prescribed quadratic formula and interference from previous learnt methods. It is important to mention that some error types occurred more than once in the same question and indifferent questions, for example, the error type, “add” when the equations have to be subtracted or vice versa.

**Students’ errors and misconceptions in Question 7**

Question 7 required students’ proper understanding of application of the elimination method in solving a system of linear simultaneous equation. Some students misinterpreted the elimination method when eliminating a variable from a system of linear equations. Some students misjudged the operations to be performed. Some students chose the reverse operation, for example, adding when it has to be subtracted or vice versa. Possibly this misunderstanding emanated from their fragile understanding of simplifying integers and manipulating signs. Their difficulties aggravated when variables in the two equations had opposites signs, (-b, b). On question 7(c) some students failed to realise that they could still obtain the same solutions whether they added or subtracted two equations.

**Student errors and misconceptions in question 7.** Consider the following system of linear equations

\[a + b = 5; \quad a - b = 7\]

(a) To eliminate \(a\), do you add or subtract the two equations? (b) To eliminate \(b\), do you add or subtract the two equations? (c) Will you obtain the same solution if you add or subtract the two equations?
Table 10: Errors and misconceptions in simultaneous equation

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misinterpreting the elimination method (procedural error)</td>
<td>7(a) Subtract</td>
<td>Add</td>
</tr>
<tr>
<td></td>
<td>7(b) Add</td>
<td>Subtract</td>
</tr>
<tr>
<td></td>
<td>7(c) Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

4.6.2. Students’ errors and misconceptions in Question 9

Question 9 in the test was meant to assess students’ understanding of solving equations. Question 9(a) was on solving linear equations. Students were asked to solve $15 - 3x = 6$. Some of the answers were incomplete solutions like $3(x - 3) = 0$ and $x = \frac{9}{3}$. Another error noted on this item was misuse of the change side – change sign rule leading to incorrect answers like $x = 7$ or $x = -3$. This supports evidence from literature. Students blindly apply the transportation of symbols technique without mathematical understanding (Kieran, 1989).

Question 9(b) was based on quadratic equations. Students were asked to solve the equation for $x$. Like in question 9(a) there were cases of incomplete solutions that were noted. There were answers like $x(x - 4) = 0$ and $x^2 - 4$. Also some students misused the change side–change sign rule when they failed to rearrange the equation and equate to zero. These students incorrectly presented solutions like $x(x + 4) = 0$. Another error was interference from previous learnt methods. Students presented incorrect answers such as $\sqrt{x} = \sqrt{4x}$, then $x = \sqrt{4x}$ or $x = (4x)^{\frac{1}{2}}$ and $\frac{x^2}{x} = \frac{4x}{x}$ then $x = 4$. This supports Tall (1991) who asserts that transition from one mental state to another may cause unstable behaviour when previous experience conflict with new ideas.

Question 9(c) required students to specifically use the quadratic formula to solve the equation $x^2 - 4x = -4$. It was clearly evident students expressed ignorance of the specified formula. They wrote $=-b \pm \sqrt{x^2-4ac} \quad 2a$. Some students disregarded this instruction and tried to factorise. Although they were able to factorise, incomplete solutions like $(x - 2)(x - 2) = 0$ were also noted. Another form of incomplete solution like $x = \frac{4 \pm \sqrt{0}}{2}$ featured after correctly using the quadratic formula, students failed to simplify the right hand side. Some students also misused the change side-change sign rule when they failed to rearrange the quadratic equation.
into standard form. They incorrectly wrote $x^2 - 4x - 4 = 0$. Nyaumwe (2004) attributes most the errors of the nature found here may be due to teachers’ pedagogical content. Methods used do not promote conceptual understanding.

Students were also required to do correct substitution into the formula. However, some students disregarded the use of essential brackets when substituting the negative value in the quadratic formula. These students presented incorrectly

$-4 \pm \sqrt{-4^2 - 4 \times 4 \times 1}
\begin{array}{c}
\end{array}
2 \times 1$

**Question 9.** Solve the following equations
(a) $15 - 3x = 6$
(b) $x^2 = 4$
(c) Use the quadratic formula to solve $x^2 - 4x = -4$

**Table 11: Students errors and misconceptions in quadratic equations**

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incomplete solution</td>
<td>9(a) 3</td>
<td>$3(x - 3) = 0; x = \frac{9}{3}$</td>
</tr>
<tr>
<td></td>
<td>9(b) 0 or 4</td>
<td>$x(x - 4) = 0; x^2 = 2^2x$</td>
</tr>
</tbody>
</table>
|                                        | 9(c) 2twice       | $(x - 2)(x - 2) = 0$
|                                        |                   | $x = \frac{4 \pm \sqrt{0}}{2}$ |
| Misuse of the change-side, change-sign rule | 9(a) 3      | -3; 7 |
|                                        | 9(b) 0 or 4       | $x(x + 4) = 0$ |
|                                        | 9(c) 2twice       | $x^2 - 4x - 4 = 0$ |
| Interference from previous methods     | 2x = 4x; x = 2x   | $\sqrt{x^2} = \sqrt{4x}; x = \sqrt{4}\frac{4}{x} x = 4$ |
| Disregarding use of brackets           | 9(c) 0 or 4       | $-4 \pm \sqrt{-4^2 - 4 \times 4 \times 1}
\begin{array}{c}
\end{array}
2 \times 1$
| Disregarding use of correct quadratic formula | 9(c) Correct formula | $(x - 2)(x - 2) = 0$ |
| Lack of knowledge of correct formula   | 9(c) Correct formula | $x = -b \pm \sqrt{b^2 - 4ac}
\begin{array}{c}
\end{array}
2a$ |
Students’ errors and misconceptions related to inequalities

In this study students experienced the most serious difficulties in solving inequalities. This conceptual category had the highest mean percentage of error. Question 10 required students’ proper understanding of solving linear inequalities. The major errors observed and noted included misuse of the change side-change sign rule, incomplete solutions, invalid distribution and changing inequalities into equations.

Table 12: Students’ errors and misconceptions in inequalities

<table>
<thead>
<tr>
<th>Type of error or possible misconception</th>
<th>Expected response</th>
<th>Incorrect response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Misuse of the change-side change-sign rule</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10(a) $x &lt; 4$;</td>
<td>$-3 &lt; 5 + 2x$</td>
<td></td>
</tr>
<tr>
<td>10(b) $x &gt; -4$;</td>
<td>$-2x &lt; 12 + 4$</td>
<td></td>
</tr>
<tr>
<td>10(c) $-3 \leq x &lt; 3$</td>
<td>$2 - 11 \leq 3x &lt; 4 - 2$</td>
<td></td>
</tr>
<tr>
<td>Changing inequalities to equations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10(a) $x &lt; 4$;</td>
<td>$x = 4$</td>
<td></td>
</tr>
<tr>
<td>10(b) $x &gt; -4$;</td>
<td>$x = -4$</td>
<td></td>
</tr>
<tr>
<td>10(c) $-3 \leq x &lt; 3$</td>
<td>$-3 = x \text{ and } x = 3$</td>
<td></td>
</tr>
<tr>
<td>Incomplete solution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10(a) $x &lt; 4$;</td>
<td>$2x &lt; 8$</td>
<td></td>
</tr>
<tr>
<td>10(b) $x &gt; -4$;</td>
<td>$-8 &lt; 2x; -2x &lt; 8$</td>
<td></td>
</tr>
<tr>
<td>10(c) $-3 \leq x &lt; 3$</td>
<td>$-3 \leq x \text{ and } x &lt; 3$</td>
<td></td>
</tr>
<tr>
<td>Invalid distribution</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10(b) $x &gt; -4$;</td>
<td>$4 - x &lt; 12$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2 - 2x &lt; 12$</td>
<td></td>
</tr>
</tbody>
</table>

Question 10 was meant to test students’ proper understanding of solving linear inequalities in one variable. Students were asked to solve the inequalities 10(a) $2x - 3 < 5$, 10(b) $2(2 - x) < 12$ and 10(c) $-11 \leq 3x - 2 < 4$. Some students misused the change side-change sign rule in 10 (a), (b) and (c) when they attempted to rearrange the inequalities.

Another major error noted was invalid distribution in question 10 (b) when they misused the distributive property in algebra. Some students failed to remove the brackets on $2(2 - x)$. Some students also changed inequalities into equations in all the three questions.
10. Solve the following inequalities.

(a) $\frac{1}{2}x > -3$

\[ \frac{1}{2}x + 3 \]

(a). Answer: \[ x = 3 \]

(b) $7 < 2x - 3$

(b). Answer: \[ x = 7 \]

(c) $3(x + 4) \geq \frac{2x}{3}$

(c). Answer: \[ x = 5 \]

(d) $7 \geq 2x - 1 > 3$

(d). Answer: \[ x = 4 \]

Some students misconstrued inequalities for equations. Some solutions were incomplete.

4.3 Research question 3

What are the sources of errors and misconceptions in the domain of school algebra?

Research evidence from this study revealed that sources of errors and misconceptions in learning algebra are content driven and pedagogically based. Content based sources relate to the nature of algebra. Students find algebra challenging because of the notations and language semantics used (Herscovics, 198C9), for example the functional notation was a problem for participants in this research. Algebra is a language of symbols and basic aspect of learning algebra is efficient use of symbols.
Despite the significance of mathematics, students find algebra naturally difficult (Witzel, Mercer & Miller, 2003), based on the responses in the students’ questionnaire where the majority of the refuted that algebra is easy to learn. Students make errors in learning algebra because the underpinning of this domain is generalisation of arithmetic and many students face difficulties in their transition from arithmetic to algebra. Smith, deSessa and Roschelle (1993) also report students experience misconceptions in algebra and Buxton, (1981) reported mathematics panic an algebra is part of the mythology of mathematics being difficult and for the selected few.

Another possible source of error and misconceptions in learning algebra from this study is the mode of lesson delivery in mathematics. The misconception emanating from the change sign change side rule could be causing misconceptions. Teachers consider transposing symbols and performing same operation on both sides as equivalent when solving equations. However, students view the two the solution processes as being distinct. For example the recurrence of the error where students disregard the use brackets when substituting negative values in the quadratic formula, the solution reflects may be indicative of the teacher as the source of errors. May be the teacher did not emphasise the point.

\[
\frac{-4 \pm \sqrt{-4^2 - 4 \times 4 \times 1}}{2 \times 1}
\]

When teachers bring instructions to the classroom they teach students who may have incorrectly learned strategies like the wrong quadratic formula where the division line did not cover the whole numerator \( x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a} \)

4. 4 Summary

The chapter discussed presentation of findings of the case study design. The findings were presented as per research question of the study in form of tables. The chapter discussed findings from the questionnaire, the tests and the interview transcriptions. Focus was mainly on students’ conceptions of algebra, kinds and sources of errors and misconceptions. Students found algebra difficult and abstract to comprehend. The prevalent categories of errors were conceptual errors in variables and procedural errors in expressions, functions, equations and inequalities.
The main sources of errors and misconceptions were content and pedagogically related. The abstract nature of algebra and methodology of lesson delivery cause errors and misconception in learning of algebra. The last chapter concludes the research by presentation of a summary of the study, conclusions and recommendations.

Chapter 5

Summary, conclusions and recommendations

5.0. Introduction.

The chapter focuses on the overview of the whole study. It highlights a summary of the first four chapters of the research. It focuses on the summary chapter by chapter. The chapter presents findings of the study as per research question. It presents recommendations based on findings of the study.

5.1. Summary of study
Chapter one focused on the background to the study. The chapter portrayed mathematics as a fundamental discipline necessary for scientific and technological advancement of a nation and algebra as a key content standard each student must understand. The chapter discussed the statement of the problem. Performance in mathematics has not improved for some time, despite its significance. Three research questions were presented. Assumptions, aims and purpose of the study were outlined. Limitations to the study and delimitations were presented in this chapter. Operational key terms were defined. It ended with a chapter summary.

Chapter two of this study covered the theoretical frame, conceptual framework and related literature review. It explained constructivism as the underpinning theoretical framework. Students construct knowledge through their personal experiences. On conceptual framework the chapter discussed the nature of algebra covering variables expressions, functions, equations and inequalities. The notion of a variable is the most important element of algebra. Algebra generalises arithmetic.

If algebra is viewed as generalised arithmetic then a variable is a pattern generaliser. If algebra is viewed as procedure for solving mathematical problems, then a variable is an unknown in equations. If algebra is the study of relationships then a variable is an argument in functions. If algebra is the study of structures then a variable is a symbolic object in expression.

The chapter discussed the kinds and sources of errors and misconceptions. Errors are computational, procedural or conceptual in nature. Sources of errors are covered in this chapter. Misconceptions originate from obstacles errors from absence of meaning. The error can be of arithmetic or procedure origin or pedagogical. Errors and misconceptions are characteristic of algebra. Students’ affective and emotional attitudes can be a source of errors. Errors may come from students’ self-generated-strategies

Secondary school students experience misconceptions in learning algebra. Students get confused when they take algebra. They equate learning algebra to manipulating symbols. Students experience difficulties in viewing algebra as generalised arithmetic. Related literature revealed that understanding the notion of a variable is fundamental to understanding algebra. Misconceptions are deeply rooted in students’ mind and are difficult to eradicate. It is recommended that prior to learning algebra, students should understand the notion of a variable. Arithmetic and algebra use the same symbols, hence results in misconceptions.
Chapter three covered the research methodology. It outlined the qualitative research paradigm and a case study design. The population, sample and systematic random sampling techniques were discussed. Research instruments used were written tests, a structured student questionnaire and student interview schedule. Data was presented in form of tables as per research question and students’ written tests, interview transcriptions and completed questionnaires constituted content analysis of data. Ethical considerations were considered.

Chapter 4 focused on the findings of the research of the case study design. The chapter presented the findings as per research question of the study. It focused on students’ conception of algebra, kinds and sources of errors. Data was presented in form of tables as per research question. Categories of errors and misconceptions were presented. Since the main focus was on students’ errors and misconceptions correct answers and non-responses were not included in data analysis. Students found algebra difficult and abstract to comprehend. Among other categories of errors, procedural and conceptual errors were the most prevalent. The main sources of errors and misconceptions were content and pedagogically related. The abstract nature of algebra and methodology of lesson delivery cause students errors and misconception in learning of algebra.

5.2. Research Findings and Conclusions

Regarding students’ understanding of algebra the research revealed that errors are common in students’ written work. The study revealed that errors and misconceptions are a result of mathematical thinking on the part of students; hence they are reasonable for students (Brodie, 2011). Students can commit errors due to a myriad of reasons ranging from a data entry to calculation errors. The student’s errors are actually natural steps to understanding Labinowich (quoted in Brooks 1993:93). It was evident from this research students have limited understanding of algebra, in particular the notion of a variable is not well grasped.

This research is consistent with Ricomini (2005), Luneta and Makonye (2010), errors and misconceptions are related but are different. In line with previous research by Li (2006) the
current study revealed that errors emanate from misconceptions the student holds. Concerning the kinds of errors and misconceptions, this study revealed that errors are either computational, algorithmical, procedural or conceptual in nature (Young & O’Shear, 1981). Conceptual errors and procedural errors were the most prevalent based on findings of this research.

Regarding sources of errors and misconceptions, this study revealed that the causes of students’ errors are complicated. The errors can be due to carelessness, not understanding at all, confusing different concepts and interference from previous experiences. Misconceptions in algebra can emanate from the abstract nature of algebra; algebra content is symbolic, failing transition from object-oriented thinking to process-oriented thinking and failing transition from arithmetic to algebra. Consistent with Tall (1991) this study revealed that transition from one mental to another may cause unstable behaviour when previous experience conflict with new ideas. The abstract nature of algebra was identified as the chief source of errors.

This study revealed that language difficulties or semantics of mathematical language can cause misconceptions in learning algebra (Barrera, Medina & Rabaynal, 2004). Consistent with previous literature, Lockhead and Mistre (1988), errors and misconceptions are deeply rooted in the minds of students and are difficult to dislodge.

Findings from this study supports previous literature from Resnick (1982) that students’ learning difficulties in algebra are attributable to concept learning. Conceptual knowledge is key to learning algebra.

This study confirms findings by Van Lehn and Jones (1993) who concur with Anderson, (1989) that gaps in conceptual knowledge lead to students using buggy incorrect procedures in solving problems in algebra. Student self-generated procedures or rules are sometimes faulty. These faulty rules have sensible origins. Consistent with Ginsburg (1977), faulty thinking in algebra is related to misconceptions in arithmetic. The results from this study concur with Lins and Kaput (2004) who argue that tradition of arithmetic then algebra cause immense students difficulties in learning algebra.
5.3. Recommendations

Consequently, teachers are encouraged to embrace the errors and engage with them rather than avoid them. This suggests teachers need to respond to students’ errors in ways that involve understanding of students’ thinking behind the error to inform teaching. Teachers should shift their minds from understanding of students’ errors as obstacles to learning mathematics to understanding errors and misconceptions as an integral to learning and teaching mathematics. Xiaobao and Yeping (2008) propound that understanding errors is a vital part of correcting them. Mathematics educators should shift from the tradition of arithmetic then algebra and an early introduction to algebraic reasoning is strongly recommended. Stephens (2008) recommends that students should develop rational thinking with number senses to assist with transition to literal symbols.

On the basis of findings from this study, the researcher strongly recommends in-service of teachers. Teachers are encouraged to organise school-based or cluster-based workshops to cross-pollinate mathematics pedagogical content and to strategise how to demystify math phobia and algebra mythology.

5.4 Summary

The fundamental intention of this study was to explore students’ conceptions of algebra, kinds of errors and misconceptions in algebra and their sources. Interviewing students provided insightful and rich information.

The qualitative data was further supplemented by the qualitative analysis. This yielded a wealth of information that could not have been tapped through other methods. In the whole the validity and the reliability of the findings were enhanced as the result of the methodology undertaken and students’ narrations.

Despite the difficulty of directly accessing students’ mathematical thinking and reasoning behind their actions, there are other methods of accessing their thinking such as interviewing. The interviewing process was quite exhaustive and demanding. Students sometimes make capricious or careless errors.

The researcher always tried to show the whole picture by relying on the data and presenting other explanations as well. Although personal perceptions, interpretations and assumptions
played a key role in identifying students’ errors and misconceptions, the researcher attempted to validate assumptions by regularly referring back to the data.

The case study approach employed in this study was equally or more important than any other empirical investigation methods in education based on controlled experimentation. Theoretically, no two individuals populations are have the same characteristics. In general, it is difficult to replicate research on human beings. However, in essence, findings in this research can safely be replicated for a population with similar or same characteristics.

The researcher articulated a number of student errors and misconceptions based on the findings of this research. The researcher also explained in detail the nature of algebra, where possible the kinds and origins of errors and misconceptions. The implications of the research were discussed with possible suggestions for classroom practice. Finally, suggestions for further research were given.

5.5 Further research

Although this research work has been a long journey, it should not end here. A number of issues have been raised during the research process. This fascinating area deserves further exploration.

Future researchers can carefully identify specific errors and misconceptions and carry further thorough analysis and careful exploration into how student errors and misconceptions in the domain of secondary school algebra impede the learning of mathematics. Further research can be done on systematic errors. Systematic errors are recurrent wrong responses methodically constructed and produced across time and space. They were not analysed because the test was not repeated to study them. They deservingly require more thorough analysis and careful exploration.
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89


*American Journal of Sociology*, 110-131

University of Bristol.


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Teachers of Mathematics.


7. APPENDICES

7.1. APPENDIX A: LETTER FROM MINISTRY OF EDUCATION
Reference: C/426/3 Masvingo
Ministry of Primary and
Secondary Education
P.O Box CY 121
Causeway
Harare

16 February 2016

Osten Ndovo
Bindura University of Science Education
P.Bag 1020
Bindura

RE: PERMISSION TO CARRY OUT RESEARCH IN MASVINGO PROVINCE:
MASVINGO DISTRICT: NDARAMA; VICTORIA AND MUCHEKE HIGH
SCHOOLS

Reference is made to your application to carry out a research in the above mentioned
schools in Masvingo Province on the research title:

"SECONDARY SCHOOL STUDENTS ERRORS AND MISCONCEPTIONS
IN LEARNING ALGEBRA"

Permission is hereby granted. However, you are required to liaise with the Provincial
Education Director Masvingo Province, who is responsible for the schools which you
want to involve in your research.

You are required to provide a copy of your final report to the Secretary for Primary and
Secondary Education by December 2016.

P. Mizawazi
DIRECTOR: Policy Planning, Research Development
For: SECRETARY FOR PRIMARY AND SECONDARY EDUCATION

cc: P.E.D. Masvingo

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7.2. APPENDIX B: LETTER TO THE SCHOOL HEADMASTER
Request to conduct research at school.

I am a second year MSc. Education (Mathematics) student in Bindura University of Science Education. My dissertation supervisor is Ms. Mutambara. For the final dissertation, I hope to conduct a research which explores secondary school students’ learning difficulties in algebra. I have your school to collect data for this study.

The purpose of this study is to examine students’ errors and misconceptions when solving algebraic tasks and to suggest remedial strategies to overcome these challenges. I intend to administer a test instrument to 65 students in Form 3 classes. The test will contain 25 short answer questions. Each test will take about one hour to answer. Later on, five students will be selected for interviews based on their responses to the test and each interview will take 20 to 30 minutes. Each interview will be audio tape recorded for later transcription.

I kindly request the participation of your school in this study by allowing me to conduct the test and interviews. You will also be given an opportunity to receive a summary of the findings. I will not use the students’ names or anything else that might identify them in their written work, oral presentations, or publications. The information remains confidential and they are free to change their minds at any time, and withdraw even if they have consented to participate. They may decline to answer specific questions. There are no known risks for assisting in this study. I will destroy the video tape recording after the research has been presented and or published, which may take up to five years after the data has been collected.

If you would like more information, please contact me at your earliest convenience to discuss the work or provide your consent to participate on 0774396142 or 0716138595 or by e-mail on ndemoosten@yahoo.com.

Thank you for your consideration.

Yours sincerely,

Osten Ndemo
Osten Ndemo
Bindura University of Science Education
P. Bag 1020
Bindura

RE: PERMISSION TO CARRY OUT RESEARCH IN MASVINGO PROVINCE:
MASVINGO DISTRICT: ZIMUTO HIGH SCHOOL.

Reference is made to your application to carry out a research in the above mentioned school in
Masvingo Province on the research title:

“SECONDARY SCHOOL STUDENTS ERRORS AND
MISCONCEPTIONS IN LEARNING ALGEBRA”

Permission is hereby granted.
A Head

H Mashava
THE HEAD
ZIMUTO HIGH SCHOOL

13 AUG 2015
PRIVATE BAG 9038
MASVINGO
TEL: 039- 264209

7.4. APPENDIX D: PARTICIPANT CONSENT FORM.
I --------------- agree to participate

(Full name)

(a) In the test  □  (Tick √)

(b) In the interview □

Participant’s Signature__________________________ Date-------------------

Appendix B: Participant Consent Form.

I □ agree to participate

(Full name)

(a) In the test  √ (Tick √)

(b) In the interview □

Participant’s Signature__________________________ Date  3/1/19
Appendix B. Participant Consent Form.

I ______________________________ agree to participate
(full name)

(a) In the test [ ] (Tick ✓) 
(b) In the interview [x] 

Participant’s Signature ______________________________ Date: 23 November 2019.

I ______________________________ agree to participate
(Full name)

(a) In the test [ ] (Tick ✓) 
(b) In the interview [x] 

Participant’s Signature ______________________________ Date 30/11/15
Appendix B. Participant Consent Form.

I __________ agree to participate
(full name)

(a) In the test □ (Tick √)

(b) In the interview □

Participant's Signature ___________________________ Date 26/09/15

Appendix B. Participant Consent Form.

I __________ agree to participate
(full name)

(a) In the test □ (Tick √)

(b) In the interview □

Participant's Signature ___________________________ Date 26/09/15
7.5. APPENDIX E: STUDENTS’ QUESTIONNAIRE.

My name is Osten Ndemo. I am a student at Bindura University of Science Education pursuing a Master of Science degree in Mathematics education. The questionnaire is part of the research study on students’ errors and misconception in learning school algebra.

Instructions

1. Please answer all questions.
2. Responses will be treated as highly confidential.
3. When given a choice put an X in the appropriate box.

Section A: Personal Information

1. Gender: Male [ ] Female [ ]
2. Age (years) [ ] 14 [ ] 15 [ ] 16 [ ] 17 [ ]
3. Status: Day-scholar [ ] Boarder [ ]

4. Grade 7 mathematics results

<table>
<thead>
<tr>
<th>Grade</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
</table>

Section B: Mark with an X to show whether the statement is True or False

<table>
<thead>
<tr>
<th>Statement</th>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. A letter can represent any number</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. The four operations are applicable to arithmetic only</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. Rules of precedence are applicable to algebra</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. Understanding arithmetic is key to understanding algebra</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Algebra is one of the most important areas of mathematics</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. Algebra generalises arithmetic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. Algebra is easy to understand</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. Algebra is interesting to learn</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thank you for your cooperation.
### APPENDIX F: STUDENTS INTERVIEW SCHEDULE

<table>
<thead>
<tr>
<th>Process</th>
<th>Interview question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Reading</td>
<td>1. Please read the question.</td>
</tr>
<tr>
<td>2. Comprehension</td>
<td>2. What does the question mean?</td>
</tr>
<tr>
<td>3. Strategy selection</td>
<td>3. How will you solve the question?</td>
</tr>
<tr>
<td>5. Encoding</td>
<td>5. Write down the answer.</td>
</tr>
<tr>
<td>6. Consolidation</td>
<td>6. What does the answer mean?</td>
</tr>
<tr>
<td>7. Verification</td>
<td>7. How do make sure your answer is correct?</td>
</tr>
<tr>
<td>8. Conflict</td>
<td>8. The interviewer can ask some conflicting questions to prove</td>
</tr>
<tr>
<td></td>
<td>Whether the student has conflict in solving process</td>
</tr>
</tbody>
</table>
7.7 APPENDIX G: PILOT-TEST.

Student Number________

This is a non-evaluative assessment. Your performance in this assessment will have no bearing on your grades or results. The assessment is designed to help you with algebra, to understand the errors you make and the misconceptions you hold as well as why you make them.

Instructions

• Answer all questions.
• Use algebra methods to solve all problems.
• Time: one hour.

1. What is a variable? Explain with an example.
Answer____________________________________________________________________

2. Bene sells \( x \) sweets and Billy sells twice as many sweets as Bene. A sweet costs 10 cents.

(a) Name a variable in this problem.
Answer____________________________________________________________________

(b) Name another variable in this problem.
Answer____________________________________________________________________

(c) Name something that is not a variable in this problem.
Answer____________________________________________________________________

3. What does \( xy \) mean? Write you answer in words.
Answer____________________________________________________________________

4. Simplify the following expressions (a) \( 3(x - 5) \) (a) Answer __________
   
(b) \( 9 + 2(5y + 11) \) (b) Answer __________
5. Factorise each expression completely
(a) \(3n + 15\)  
(b) \(10x^3 - 10x\)

6. Given that \(f(x) = 4 - x\), find
(a) \(f(0)\)
(b) \(f(-2)\)
(c) \(x\) when \(f(x) = 0\)

7. Solve
(a) \(2x + 3 = x\)
(b) \(x^2 - 4 = 0\)
(c) \(x^2 - 4x = -4\)

8. Consider the following system of linear equations \(x + y = 3, \quad x - y = 2\)
   (a) To eliminate \(x\), do you add or subtract the two equations?  
      (a) Answer___________
   (b) To eliminate \(y\), do you add or subtract the two equations?  
      (b) Answer___________

9. (a) Solve the following system of linear equations \(2a + b = 2, \quad 3a - b = 3\)
   (a) Answer \(a = \)___________ \(b = \)___________
   (b) Solve the following system of equations using substitution method
      \(x + y = 3, \quad y = 2x + 3\)  
      (b) Answer___________
      \(x\)___________ \(y\)___________

10. Solve the following inequalities
    (a) \(\frac{1}{2}x > -3\)  
        (a) Answer___________
    (b) \(7 < 2x - 3\)  
        (b) Answer___________
    (c) \(3(x + 4) \geq \frac{2x}{3}\)  
        (c) Answer___________
    (d) \(7 \geq 2x - 1 > 3\)  
        (d) Answer___________

End of question paper. Thank you.
7.8. APPENDIX H: MAIN-TEST.

Student Number ________

This is a non-evaluative assessment. Your performance in this assessment will have no bearing on your grades or results. The assessment is designed to help you with algebra, to understand the errors you make and the misconceptions you hold and why you make them.

Instructions

- Answer all questions.
- Use algebra methods to solve all problems.
- Time: one hour.

1. Bene sells $x$ sweets and Billy sells twice as many sweets as Bene. A sweet costs 10 cents.

(a) Name a variable in this problem.

Answer ______________________________________________________________________

(b) Name something that is not a variable in this problem.

Answer ______________________________________________________________________

2. What does $xy$ mean? Write your answer in words.

Answer ______________________________________________________________________

3. Apples cost $a$ cents and bananas cost $b$ cents. If 3 apples and 2 bananas are sold, what does $3a + 2b$ represent? Answer ______________________________________________________________________

4. (a) Multiply $x + 3$ by 2

Answer ________________

(b) Add $4y$ to 3

Answer ________________

(c) Subtract $2b$ from 7

Answer ________________

5. (a) Which is larger $\frac{1}{n}$ or $\frac{1}{n+1}$, when $n$ is a natural number? (a) Answer ________________

(b) Which is larger $y$ or $x$ in $y = 2x + 3$ (b) Answer ________________
6. Simplify (a) $7 + 3x$  
   (a) Answer________________
   (b) $p + p - (2c + 2p)$  
   (b) Answer________________
   (c) $x \left( \frac{a}{b} \right)$  
   (c) Answer________________
   (d) $\frac{ax + bx}{x+d}$  
   (d) Answer________________

7. Consider the following system of linear equations $a + b = 5$, $a - b = 7$

   (a) To eliminate $a$, do you add or subtract the two equations? (a) Answer________________
   (b) To eliminate $b$, do you add or subtract the two equations? (b) Answer________________
   (c) Will you obtain the same solution if you add or subtract the two equations? (c) Answer________________

8. Given that $f(x) = x^2 - 1$, find

   (a) $f(0)$  
   (a) Answer________________
   (b) $f(-2)$  
   (b) Answer________________
   (c) $x$ when $f(x) = 0$  
   (c) Answer________________

9. Solve the following equations (a) $15 - 3x = 6$  
   (a) Answer________________
   (b) $x^2 = 4$  
   (b) Answer________________
   (c) Use the quadratic formula to solve $x^2 - 4x = -4$  
   (c) Answer________________

10. Solve the following inequalities (a) $2x - 3 < 5$  
    (a) Answer________________
    (b) $2(2 - x) < 1$  
    (b) Answer________________
    (c) $-11 \leq 3x - 2$  
    (c) Answer________________

End of question paper. Thank you
## 8. TABLES

Table 13: Composition of questions in five different conceptual categories in the test.

<table>
<thead>
<tr>
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